ORTHOGONAL POLYNOMIAL SOLUTIONS OF A CLASS OF SIXTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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TABLE OF CONTENTS

INTRODUCTION

Certain classes of linear differential equations have as particular solutions sets of polynomials which possess the property of orthogonality, with respect to "weight" functions, in given intervals of the independent variable. The polynomials of Hermite, Laguerre, and Jacobi (which include the Legendre polynomials as a special case) are familiar examples of sets of orthogonal solutions of a class of linear equations of the second order. The expansion of an arbitrary function in a series of orthogonal polynomial solutions of a given differential equation has proved a useful tool in analysis.

In this paper a class of linear differential equations of the sixth order is considered. First, conditions are determined under which polynomial solutions of the general equation of the class exist, and then further conditions are developed which insure that these polynomial solutions will be orthogonal in any fundamental interval. Formal expansion of an arbitrary function in a series of the solutions is then demonstrated. Next, the restrictions imposed on the forms of the weight function and the coefficients of the equation in their satisfaction of the orthogonality conditions by the choice of a particular fundamental interval are obtained in separate treatments of the finite, semi-infinite, and infinite intervals. An example of each case is presented to demonstrate the application of the conditions obtained.

CHAPTER I

POLYNOMIAL SOLUTIONS

The Differential Equation

Consider the differential equation of the sixth order

$$(1.1)\quad p(x)y_n^{vi} + q(x)y_n^{v} + r(x)y_n^{iv} + s(x)y_n''' + t(x)y_n'' + u(x)y_n' + \lambda_n v(x)y_n = 0,$$

where the coefficients p, q, r, s, t, u, and v, none of which vanish identically, are assumed to be real polynomials in the real variable x, and λ_n is a polynomial in n.

Form of Solutions

Polynomials of the form

$$(1.2)\quad y_n = \sum_{j=0}^{n} a_{jn}x^{n-j}, \quad a_{nn} \neq 0, \quad n = 0,1,2, \cdots,$$

are desired as solutions of the differential equation (1.1). A solution set y_n, n = 0,1,2, ..., can thus be written as

$$(1.3)\quad \begin{cases} y_0 = a_{00}, & a_{00} \neq 0 \\ y_1 = a_{01}x + a_{11}, & a_{01} \neq 0 \\ y_2 = a_{02}x^2 + a_{12}x + a_{22}, & a_{02} \neq 0 \\ \quad\cdots & \\ y_n = a_{0n}x^n + a_{1n}x^{n-1} + \cdots + a_{nn} & \\ \quad\cdots & \end{cases}$$

The requirement of the members of (1.3) to be solutions of the differential equation (1.1) will impose certain restrictions on the coefficients p, q, r, s, t, u, and v of equation (1.1).

<u>The solution $y_0 = a_{00}$.</u> If $y_0 = a_{00}$ is to be a solution of equation (1.1), then

(1.4) $\lambda_0 v(x) a_{00} = 0.$

Since $v(x) \neq 0$ and $a_{00} \neq 0$, then $\lambda_0 = 0$. This requirement imposes no restrictions on p, q, r, s, t, u, and v.

<u>The solution $y_1 = a_{01}x + a_{11}$.</u> If $y_1 = a_{01}x + a_{11}$ is to be a solution of equation (1.1), then

(1.5) $u(x)a_{01} + \lambda_1 v(x)(a_{01}x + a_{11}) = 0.$

Since $a_{01} \neq 0$, then $\lambda_1 \neq 0$, as $u(x)$ would vanish identically for $\lambda_1 = 0$. Division of equation (1.5) by $v(x)$ yields

(1.6) $U(x) = \dfrac{u(x)}{v(x)} = -\dfrac{\lambda_1}{a_{01}}(a_{01}x + a_{11}) \neq 0.$

Hence $U(x)$ is a polynomial of degree one, and equation (1.1) becomes

(1.7) $p(x)y_n^{vi} + q(x)y_n^{v} + r(x)y_n^{iv} + s(x)y_n^{'''} + t(x)y_n^{''} + v(x)U(x)y_n^{'} + \lambda_n v(x)y_n = 0.$

<u>The solution $y_2 = a_{02}x^2 + a_{12}x + a_{22}$.</u> If $y_2 = a_{02}x^2 + a_{12}x + a_{22}$ is to be a solution of equation (1.7), then

(1.8) $2t(x)a_{02} + t(x)U(x)(2a_{02}x + a_{12}) + \lambda_2 v(x)(a_{02}x^2 + a_{12}x + a_{22}) = 0.$

Division of equation (1.8) by $v(x)$ yields

$$(1.9)\quad T(x) = \frac{t(x)}{v(x)}$$

$$= -\frac{1}{2a_{02}}[U(x)(2a_{02}x + a_{12}) + \lambda_2(a_{02}x^2 + a_{12}x + a_{22})] \neq 0.$$

Hence, $T(x)$ is a polynomial of degree two, at most. Equation (1.7) now becomes

$$(1.10)\quad p(x)y_n^{vi} + q(x)y_n^{v} + r(x)y_n^{iv} + s(x)y_n''' + v(x)T(x)y_n''$$

$$+ v(x)U(x)y_n' + \lambda_n v(x)y_n = 0.$$

The solution $y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}$. If this solution is to satisfy equation (1.10), then

$$(1.11)\quad 6s(x)a_{03} + v(x)T(x)(6a_{03}x + 2a_{13})$$

$$+ v(x)U(x)(3a_{03}x^2 + 2a_{13}x + a_{23})$$

$$+ \lambda_3 v(x)(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}) \equiv 0.$$

Division of equation (1.11) by $v(x)$ yields

$$(1.12)\quad S(x) = \frac{s(x)}{v(x)}$$

$$= -\frac{1}{6a_{03}}[T(x)(6a_{03}x + 2a_{13}) + U(x)(3a_{03}x^2 + 2a_{13}x + a_{23})$$

$$+ \lambda_3(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33})] \neq 0.$$

Hence, $S(x)$ is a polynomial of degree three, at most. Equation (1.10) can now be written

$$(1.13)\quad p(x)y_n^{vi} + q(x)y_n^{v} + r(x)y_n^{iv} + v(x)S(x)y_n^{'''} + v(x)T(x)y_n^{''} + v(x)U(x)y_n^{'} + \lambda_n v(x)y_n = 0.$$

The solution $y_4 = a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44}$. If this solution is to satisfy equation (1.13), then

$$(1.14)\quad 24r(x)a_{04} + v(x)S(x)(24a_{04}x + 6a_{14}) + v(x)T(x)(12a_{04}x^2 + 6a_{14}x + 2a_{24}) + v(x)U(x)(4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34}) + \lambda_4 v(x)(a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44}) = 0.$$

Division of equation (1.14) by $v(x)$ yields

$$(1.15)\quad R(x) = \frac{r(x)}{v(x)} = -\frac{1}{24a_{04}}[S(x)(24a_{04}x + 6a_{14}) + T(x)(12a_{04}x^2 + 6a_{14}x + 2a_{24}) + U(x)(4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34}) + \lambda_4(a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44})] \neq 0.$$

Hence $R(x)$ is a polynomial of degree four, at most. Equation (1.13) now becomes

$$(1.16)\quad p(x)y_n^{vi} + q(x)y_n^{v} + v(x)R(x)y_n^{iv} + v(x)S(x)y_n^{'''} + v(x)T(x)y_n^{''} + v(x)U(x)y_n^{'} + \lambda_n v(x)y_n = 0.$$

The solution $y_5 = a_{05}x^5 + a_{15}x^4 + a_{25}x^3 + a_{35}x^2 + a_{45}x + a_{55}$.

If this solution is to satisfy equation (1.16), then

$$(1.17)\quad \begin{aligned} 120q(x)a_{05} &+ v(x)R(x)(120a_{05}x + 24a_{15}) \\ &+ v(x)S(x)(60a_{05}x^2 + 24a_{15}x + 6a_{25}) \\ &+ v(x)T(x)(20a_{05}x^3 + 12a_{15}x^2 + 6a_{25}x + 2a_{35}) \\ &+ v(x)U(x)(5a_{05}x^4 + 4a_{15}x^3 + 3a_{25}x^2 + 2a_{35}x + a_{45}) \\ &+ \lambda_5 v(x)(a_{05}x^5 + a_{15}x^4 + a_{25}x^3 + a_{35}x^2 + a_{45}x + a_{55}) = 0 \end{aligned}$$

Division of equation (1.17) by $v(x)$ yields

$$(1.18)\quad \begin{aligned} Q(x) &= \frac{q(x)}{v(x)} \\ &= -\frac{1}{120a_{05}}[R(x)(120a_{05}x + 24a_{15}) \\ &\quad + S(x)(60a_{05}x^2 + 24a_{15}x + 6a_{25}) \\ &\quad + T(x)(20a_{05}x^3 + 12a_{15}x^2 + 6a_{25}x + 2a_{35}) \\ &\quad + U(x)(5a_{05}x^4 + 4a_{15}x^3 + 3a_{25}x^2 + 2a_{35}x + a_{45}) \\ &\quad + \lambda_5(a_{a5}x^5 + a_{15}x^4 + a_{25}x^3 + a_{35}x^2 + a_{45}x + a_{55})] \neq 0 \end{aligned}$$

Hence, $Q(x)$ is a polynomial of degree five, at most. Equation (1.16) can now be written

$$(1.19)\quad \begin{aligned} g(x)y_n^{vi} &+ v(x)Q(x)y_n^{v} + v(x)R(x)y_n^{iv} + v(x)S(x)y_n''' + v(x)T(x)y_n'' \\ &+ v(x)U(x)y_n' + \lambda_n v(x)y_n = 0. \end{aligned}$$

The solution $y_6 = a_{06}x^6 + a_{16}x^5 + a_{26}x^4 + a_{36}x^3 + a_{46}x^2 + a_{56}x + a_{66}$. If this solution is to satisfy equation (1.19), then

$$(1.20)\quad \begin{aligned} &720p(x)a_{06} + v(x)Q(x)(720a_{06}x + 120a_{16}) \\ &+ v(x)R(x)(360a_{06}x^2 + 120a_{16}x + 24a_{26}) \\ &+ v(x)S(x)(120a_{06}x^3 + 60a_{16}x^2 + 24a_{26}x + 6a_{36}) \\ &+ v(x)T(x)(30a_{06}x^4 + 20a_{16}x^3 + 12a_{26}x^2 + 6a_{36}x + 2a_{46}) \\ &+ v(x)U(x)(6a_{06}x^5 + 5a_{16}x^4 + 4a_{26}x^3 + 3a_{36}x^2 + 2a_{46}x + a_{56}) \\ &+ \lambda_6 v(x)(a_{06}x^6 + a_{16}x^5 + a_{26}x^4 + a_{36}x^3 + a_{46}x^2 + a_{56}x + a_{66}) = 0. \end{aligned}$$

Division of equation (1.20) by $v(x)$ yields

$$(1.21)\quad \begin{aligned} P(x) &= \frac{p(x)}{v(x)} \\ &= -\frac{1}{720a_{06}}[Q(x)(720a_{06}x + 120a_{16}) \\ &+ R(x)(360a_{06}x^2 + 120a_{16}x + 24a_{26}) \\ &+ S(x)(120a_{06}x^3 + 60a_{16}x^2 + 24a_{26}x + 6a_{36}) \\ &+ T(x)(30a_{06}x^4 + 20a_{16}x^3 + 12a_{26}x^2 + 6a_{36}x + 2a_{46}) \\ &+ U(x)(6a_{06}x^5 + 5a_{16}x^4 + 4a_{26}x^3 + 3a_{36}x^2 + 2a_{46}x + a_{56}) \\ &+ \lambda_6(a_{06}x^6 + a_{16}x^5 + a_{26}x^4 + a_{36}x^3 + a_{46}x^2 + a_{56}x + a_{66})] \neq 0. \end{aligned}$$

Hence $P(x)$ is a polynomial of degree six, at most. Equation (1.19) now becomes

$$(1.22)\quad \begin{aligned} &v(x)P(x)y_n^{vi} + v(x)Q(x)y_n^{v} + v(x)R(x)y_n^{iv} + v(x)S(x)y_n''' \\ &+ v(x)T(x)y_n'' + v(x)U(x)y_n' + \lambda_n v(x)y_n = 0. \end{aligned}$$

Division of equation (1.22) by $v(x)$ yields

$$(1.23)\quad P(x)y_n^{vi} + Q(x)y_n^{v} + R(x)y_n^{iv} + S(x)y_n''' + T(x)y_n'' + U(x)y_n' + \lambda_n y_n = 0.$$

<u>The solution $y_k = a_{ok}x^k + a_{1k}x^{k-1} + \cdots + a_{kk}$, $k = 0, 1, 2, \cdots$.</u>

If this solution is to satisfy equation (1.23), then

$$\begin{aligned}(1.24)\quad & P(x)[k(k-1)(k-2)(k-3)(k-4)(k-5)a_{ok}x^{k-6} + \cdots + 720a_{(k-6)k}] \\ & + Q(x)[k(k-1)(k-2)(k-3)(k-4)a_{ok}x^{k-5} + \cdots + 120a_{(k-5)k}] \\ & + R(x)[k(k-1)(k-2)(k-3)a_{ok}x^{k-4} + \cdots + 24a_{(k-4)k}] \\ & + S(x)[k(k-1)(k-2)a_{ok}x^{k-3} + \cdots + 6a_{(k-3)k}] \\ & + T(x)[k(k-1)a_{ok}x^{k-2} + \cdots + 2a_{(k-2)k}] \\ & + U(x)[ka_{ok}x^{k-1} + \cdots + a_{(k-1)k}] \\ & + \lambda_k[a_{ok}x^k + a_{1k}x^{k-1} + a_{2k}x^{k-2} + \cdots + a_{kk}] = 0.\end{aligned}$$

The left member of equation (1.24) is thus a polynomial of degree k, at most. By a proper choice of the coefficients a_{jk}, $j = 0, 1, 2, \cdots k$, and λ_k, the left member can be made to vanish identically.

<u>Summary</u>

The differential equation (1.23) has polynomial solutions of form of equation (1.2) if $P(x)$ is a polynomial of degree six, at most; $Q(x)$ is a polynomial of degree five, at most; $R(x)$ is a polynomial of degree four, at most; $S(x)$ is a polynomial of degree three, at most; $T(x)$ is a polynomial of degree two, at most; $U(x)$ is a polynomial of degree one, and $\lambda_o = 0$.

CHAPTER II

ORTHOGONALITY OF THE SOLUTION SET

Derivation of Conditions

In the preceding chapter conditions on the polynomial coefficients P, Q, R, S, T, and U of the differential equation (1.23) have been established such that a set of polynomial solutions of the form of equation (1.2) exists for the differential equation (1.23).

A set of conditions will now be derived under which the solution set $\{y_n\}$, $n = 0,1,2,\ldots$, will form an orthogonal system, with respect to a weight function $w(x)$, over a fundamental interval $[\alpha,\beta]$.

Formation of the Basic Equation

Consider the system of equations:

$$(2.1)\qquad Py_n^{vi} + Qy_n^{v} + Ry_n^{iv} + Sy_n''' + Ty_n'' + Uy_n' + \lambda_n y_n = 0,$$

$$(2.2)\qquad Py_m^{vi} + Qy_m^{v} + Ry_m^{iv} + Sy_m''' + Ty_m'' + Uy_m' + \lambda_m y_m = 0,$$

where $n \neq m$.

Multiplication of equations (2.1) and (2.2) by wy_m and $-wy_n$, respectively, and addition of the resulting equations yields

$$(2.3)\qquad wP(y_n^{vi}y_m - y_m^{vi}y_n) + wQ(y_n^{v}y_m - y_m^{v}y_n) + wR(y_n^{iv}y_m - y_m^{iv}y_n)$$
$$+ wS(y_n'''y_m - y_m'''y_n) + wT(y_n''y_m - y_m''y_n) + wU(y_n'y_m - y_m'y_n)$$
$$+ w(\lambda_n - \lambda_m)y_n y_m = 0, \quad n \neq m.$$

Let

$$(2.4)\quad G = y_n'y_m - y_m'y_n,$$

so that

$$(2.5)\quad G' = y_n''y_m - y_m''y_n,$$

$$(2.6)\quad G'' = (y_n'''y_m - y_m'''y_n) + (y_n''y_m' - y_m''y_n'),$$

$$(2.7)\quad G''' = (y_n^{iv}y_m - y_m^{iv}y_n) + 2(y_n'''y_m' - y_m'''y_n'),$$

$$(2.8)\quad G^{iv} = (y_n^{v}y_m - y_m^{v}y_n) + 3(y_n^{iv}y_m' - y_m^{iv}y_n') + 2(y_n'''y_m'' - y_m'''y_n''),$$

$$(2.9)\quad G^{v} = (y_n^{vi}y_m - y_m^{vi}y_n) + 4(y_n^{v}y_m' - y_m^{v}y_n') + 5(y_n^{iv}y_m'' - y_m^{iv}y_n'').$$

Also let

$$(2.10)\quad H = y_n''y_m' - y_m''y_n',$$

so that

$$(2.11)\quad H' = (y_n'''y_m' - y_m'''y_n',$$

$$(2.12)\quad H'' = (y_n^{iv}y_m' - y_m^{iv}y_n') + (y_n'''y_m'' - y_m'''y_n''),$$

$$(2.13)\quad H''' = (y_n^{v}y_m' - y_m^{v}y_n') + 2(y_n^{iv}y_m'' - y_m^{iv}y_n'')$$

Also let

$$(2.14)\quad J = y_n'''y_m'' - y_m'''y_n'',$$

so that

$$(2.15)\quad J' = y_n^{iv}y_m'' - y_m^{iv}y_n''$$

From equations (2.6) through (2.15) it is clear that

$$(2.16)\quad G'' - H = y_n''' y_m - y_m''' y_n,$$

$$(2.17)\quad G''' - 2H' = y_n^{iv} y_m - y_m^{iv} y_n,$$

$$(2.18)\quad H''' - 2J' = y_n^{v} y_m' - y_m^{v} y_n',$$

$$(2.19)\quad G^{iv} - 3H'' + J = y_n^{v} y_m - y_m^{v} y_n,$$

$$(2.20)\quad G^{v} - 4H''' + 5J' = y_n^{vi} y_m - y_m^{vi} y_n$$

Substitution of equations (2.16) through (2.20) into equation (2.3) yields

$$wP(G^{v} - 4H''' + 5J') + wQ(G^{iv} - 3H'' + J) + wR(G''' - 2H')$$

$$+ wS(G'' - H) + wTG' + wUG + (\lambda_n - \lambda_m) w y_n y_m = 0, \quad n \neq m,$$

or more simply,

$$(2.21)\quad wPG^{v} + wQG^{iv} + wRG''' + wSG'' + wTG' + wUG$$

$$- 4wPH''' - 3wQH'' - 2wRH' - wSH + 5wPJ' + wQJ$$

$$+ (\lambda_n - \lambda_m) w y_n y_m = 0, \quad n \neq m.$$

Equation (2.21) is to be known as the basic equation for orthogonality

<u>Derivative Form of the Basic Equation</u>

Consider the following relationships:

$$(2.22)\quad wPG^{v} + wQG^{iv} + wRG''' + wSG'' + wTG'$$

$$= (wPG^{iv} + wQG''' + wRG'' + wSG' + wTG)'$$

$$- (wP)'G^{iv} - (wQ)'G''' - (wR)'G'' - (wS)'G' - (wT)'G,$$

$$(2.23)\quad -(wP)'G^{iv} - (wQ)'G''' - (wR)'G'' - (wS)'G'$$
$$= [-(wP)'G''' - (wQ)'G'' - (wR)'G' - (wS)'G]' + (wP)''G'''$$
$$+ (wQ)''G'' + (wR)''G' + (wS)''G,$$

$$(2.24)\quad (wP)''G''' + (wQ)''G'' + (wR)''G'$$
$$= [(wP)''G'' + (wQ)''G' + (wR)''G]' - (wP)'''G'' - (wQ)'''G'$$
$$- (wR)'''G,$$

$$(2.25)\quad -(wP)'''G'' - (wQ)'''G' = [-(wP)'''G' - (wQ)'''G]'$$
$$+ (wP)^{iv}G' + (wQ)^{iv}G,$$

$$(2.26)\quad (wP)^{iv}G' = [(wP)^{iv}G]' - (wP)^{v}G,$$

$$(2.27)\quad -4wPH''' - 3wQH'' - 2wRH' - wSH = (-4wPH'' - 3wQH' - 2wRH)'$$
$$+ 4(wP)'H'' + 3(wQ)'H' + [2(wR)' - wS]H,$$

$$(2.28)\quad 4(wP)'H'' + 3(wQ)'H' = [4(wP)'H' + 3(wQ)'H] - 4(wP)''H' - 3(wQ)''H,$$

$$(2.29)\quad -4(wP)''H' = [-4(wP)''H]' + 4(wP)'''H,$$

$$(2.30)\quad 3wPJ' + wQJ = 3(wPJ)' + [wQ - 3(wP)']J.$$

Substitution of equations (2.22) through (2.30) into equation (2.21) yields

$$(2.31)\quad wPG^{iv} + [wQ - (wP)']G''' + [wR - (wQ)' + (wP)'']G''$$
$$+ [wS - (wR)' + (wQ)'' - (wP)''']G'$$

$$+ [wT - (wS)' + (wR)'' - (wQ)''' + (wP)^{iv}]G]'$$

$$+ [wU - (wT)' + (wS)'' - (wR)''' + (wQ)^{iv} - (wP)^{v}]G$$

$$+ \{-4wPH'' + [4(wP)' - 3wQ]H' + [-4(wP)'' + 3(wQ)' - 2wR]H\}'$$

$$+ [4(wP)''' - 3(wQ)'' + 2(wR)' - wS]H$$

$$+ 3(wPJ)' + [wQ - 3(wP)']J$$

$$+ (\lambda_n - \lambda_m)\, w y_n y_m = 0, \quad n \neq m.$$

Equation (2.31) is the derivative form of the basic equation (2.21).

<u>Integration of the Derivative Form of the Basic Equation</u>

Let equation (2.31) be integrated with respect to x from $x = \alpha$ to $x = \beta$. That is

$$(2.32) \quad \int_\alpha^\beta \{wPG^{iv} + [wQ - (wP)']G''' + [wR - (wQ)' + (wP)'']G''$$

$$+ [wS - (wR)' + (wQ)'' - (wP)''']G'$$

$$+ [wT - (wS)' + (wR)'' - (wQ)''' + (wP)^{iv}]G\}'\,dx$$

$$+ \int_\alpha^\beta [wU - (wT)' + (wS)'' - (wR)''' + (wQ)^{iv} - (wP)^{v}]G\,dx$$

$$+ \int_\alpha^\beta \{-4wPH'' + [4(wP)' - 3wQ]H' + [-4(wP)'' + 3(wQ)' - 2wR]H\}'\,dx$$

$$+ \int_\alpha^\beta [4(wP)''' - 3(wQ)'' + 2(wR)' - (wS)]H\,dx + \int_\alpha^\beta [wQ - 3(wP)']J\,dx$$

$$+ \int_\alpha^\beta (\lambda_n - \lambda_m) w y_n y_m\,dx = 0, \quad n \neq m.$$

Performance of the indicated integration in equation (2.32) yields

$$(2.33)\quad wPG^{iv}\Big]_{\alpha}^{\beta} + [wQ - (wP)']G'''\Big]_{\alpha}^{\beta} + [wR - (wQ)' + (wP)'']G''\Big]_{\alpha}^{\beta}$$

$$+ [wS - (wR)' + (wQ)'' - (wP)''']G'\Big]_{\alpha}^{\beta}$$

$$+ [wT - (wS)' + (wR)'' - (wQ)''' + (wP)^{iv}]G\Big]_{\alpha}^{\beta}$$

$$+ \int_{\alpha}^{\beta} [wU - (wT)' + (wS)'' - (wR)''' + (wQ)^{iv} - (wP)^{v}]G dx$$

$$- [4(wP)' - 3wQ]H'\Big]_{\alpha}^{\beta} + [-4(wP)'' + 3(wQ)' - 2wR]H\Big]_{\alpha}^{\beta}$$

$$+ \int_{\alpha}^{\beta} [4(wP)''' - 3(wQ)'' + 2(wR)' - wS]H dx + 3wPJ\Big]_{\alpha}^{\beta}$$

$$+ \int_{\alpha}^{\beta} [wQ - 3(wP)']J dx + (\lambda_n - \lambda_m)\int_{\alpha}^{\beta} w y_n y_m dx = 0, \quad n \neq m$$

<u>Placement of Conditions on the Integrated Basic Equation</u>

The desired result is

$$\int_{\alpha}^{\beta} w y_n y_m dx = 0, \quad n \neq m,$$

which is the usual definition of an orthogonal system. To achieve this end, the following conditions will be chosen:

$$(2.34)\quad (wP) = 0 \ \text{ at } \ x = \alpha \ \text{ and } \ x = \beta,$$

$$(2.35)\quad wQ - (wP)' = 0 \ \text{ at } \ x = \alpha \ \text{ and } \ x = \beta,$$

(2.36) $wR - (wQ)' + (wP)'' = 0$ at $x = \alpha$ and $x = \beta$,

(2.37) $wS - (wR)' + (wQ)'' - (wP)''' = 0$ at $x = \alpha$ and $x = \beta$,

(2.38) $wT - (wS)' + (wR)'' - (wQ)''' + (wP)^{iv} = 0$ at $x = \alpha$ and $x = \beta$,

(2.39) $wU - (wT)' + (wS)'' - (wR)''' + (wQ)^{iv} - (wP)^{v} \equiv 0$,

(2.40) $4(wP)' - 3wQ = 0$ at $x = \alpha$ and $x = \beta$,

(2.41) $-4(wP)'' + 3(wQ)' - 2wR = 0$ at $x = \alpha$ and $x = \beta$,

(2.42) $4(wP)''' - 3(wQ)'' + 2(wR)' - wS \equiv 0$,

(2.43) $wQ - 3(wP)' \equiv 0$.

Under conditions (2.34) through (2.43), the integral between $x = \alpha$ and $x = \beta$ of the derivative form (2.31) of the basic equation reduces to

$$(\lambda_n - \lambda_m)\int_\alpha^\beta wy_n y_m = 0, \quad n \neq m.$$

Now if the parameter values λ_n are all distinct,

$$(2.44) \quad \int_\alpha^\beta wy_n y_m dx = 0, \quad n \neq m.$$

Hence equations (2.34) through (2.43) are sufficient conditions of orthogonality of the solution set (1.2) of differential equation (1.23) over $[\alpha,\beta]$. This set of conditions may be simplified. Substitution of equation (2.43) into equation (2.40) yields

(2.45) $(WP)' = 0$ at $x = \alpha$ and $x = \beta$.

Addition of equation (2.41) to twice equation (2.36) yields

(2.46) $(WQ)' = 2(WP)''$ at $x = \alpha$ and $x = \beta$.

Differentiation of equation (2.43) yields

(2.47) $(WQ)' = 3(WP)''$.

Substitution of equation (2.47) into equation (2.46) yields

(2.48) $(WP)'' = 0$ at $x = \alpha$ and $x = \beta$.

Substitution of equation (2.48) into equation (2.36) yields

(2.49) $WR - (WP)'' = 0$ at $x = \alpha$ and $x = \beta$.

Differentiation of equation (2.47) yields

(2.50) $(WQ)'' = 3(WP)'''$.

Substitution of equation (2.50) into equation (2.42) yields

(2.51) $WS = 2(WR)' - 5(WP)'''$.

Substitution of equation (2.50) into equation (2.37) yields

(2.52) $-WS - 2(WP)''' = -(WR)'$ at $x = \alpha$ and $x = \beta$.

Addition of equation (2.51) to equation (2.52) yields

(2.53) $(WR)' - 3(WP)''' = 0$ at $x = \alpha$ and $x = \beta$.

Differentiation of equations (2.50) and (2.51) yields, respectively,

$$(2.54)\quad (wQ)''' = 3(wP)^{iv},$$

$$(2.55)\quad (wS)' = 2(wR)'' - 5(wP)^{iv}.$$

Substitution of equations (2.54) and (2.55) into equation (2.38) yields

$$(2.56)\quad wT - (wR)'' + 3(wP)^{iv} = 0 \quad \text{at} \quad x = \alpha \quad \text{and} \quad x = \beta$$

Differentiation of equations (2.54) and (2.55) yields, respectively,

$$(2.57)\quad (wQ)^{iv} = 3(wP)^{v},$$

$$(2.58)\quad (wS)'' = 2(wR)''' - 5(wP)^{v}$$

Substitution of equations (2.57) and (2.58) into equation (2.38) yields

$$(2.59)\quad wU = (wT)' - (wR)''' + 3(wP)^{v}.$$

Equations (2.54), (2.43), (2.45), (2.48), (2.49), (2.51), (2.53), (2.56), and (2.59) are independent, in the sense that no one of them can be derived from any combination of the others, and they constitute a complete set of orthogonality conditions for the solution set (1.2) of differential equation (1.25) over $[\alpha, \beta]$, since from these nine equations the ten orthogonality conditions represented by equations (2.34) through (2.43) may be obtained.

Summary

It has been established that under the nine orthogonality conditions

(i) $wP = 0$ at $x = \alpha$ and $x = \beta$

(ii) $(wP)' = 0$ at $x = \alpha$ and $x = \beta$

(iii) $(wP)'' = 0$ at $x = \alpha$ and $x = \beta$

(iv) $wR - (wP)'' = 0$ at $x = \alpha$ and $x = \beta$

(v) $(wR)' - 3(wP)''' = 0$ at $x = \alpha$ and $x = \beta$

(vi) $wT - (wR)'' + 5(wP)^{iv} = 0$ at $x = \alpha$ and $x = \beta$

(vii) $wS = 2(wR)' - 5(wP)'''$

(viii) $wU = (wT)' - (wR)''' + 5(wP)^{v}$

(ix) $wQ = 3(wP)'$

the solution set $\{y_n\}$, $n = 0,1,2, \cdots$, represented by equation (1.2) of the differential equation (1.23) will form an orthogonal system, with respect to the weight function $w(x)$, over a fundamental interval $[\alpha,\beta]$.

This may be more explicitly expressed by stating that if $w(x)$ is assumed to be non-negative in the interval $[\alpha,\beta]$, then the set $\{w^{1/2}y_n\}$, $n = 0,1,2, \cdots$, forms an orthogonal system over the fundamental interval $[\alpha,\beta]$.

CHAPTER III

EXPANSION OF AN ARBITRARY FUNCTION IN SERIES

Assume that an arbitrary function $f(x)$ can be expanded as

$$(3.1) \quad f(x) = a_0 y_0 + a_1 y_1 + \cdots + a_n y_n + \cdots + a_m y_m + \cdots ,$$

where the a_k are constants to be determined and the y_k are solutions of equation (1.23). The determination of the constants a_k, $k = 0,1,2, \cdots$, will make it possible to formally expand $f(x)$ as a series in the polynomial solutions of differential equation (1.23).

Multiply both sides of equation (3.1) by wy_n to obtain

$$(3.2) \quad wf(x)y_n = a_0 wy_0 y_n + \cdots + a_n wy_n^2 + \cdots + a_m wy_n y_m + \cdots .$$

Integration of equation (3.2) with respect to x over the fundamental interval of orthogonality $[\alpha,\beta]$ yields

$$(3.3) \quad \int_\alpha^\beta wf(x)y_n dx = a_0 \int_\alpha^\beta wy_0 y_n dx + a_1 \int_\alpha^\beta wy_1 y_n dx + \cdots$$
$$+ a_n \int_\alpha^\beta wy_n^2 dx + \cdots$$
$$+ a_m \int_\alpha^\beta wy_n y_m dx + \cdots .$$

Equation (2.44) may now be utilized, so that every term of the right member of equation (3.3), with the exception of the integrated square, vanishes. Thus

$$(3.4)\qquad \int_{\alpha}^{\beta} wf(x)y_n dx = a_n \int_{\alpha}^{\beta} wy_n^2 dx.$$

Each a_n, $n = 0,1,2, \cdots$, can now be uniquely determined, subject only to the integrability of the expressions in equation (3.4).

Hence, an arbitrary function of x can be formally expanded as a series in the solution set $\{y_n\}$, $n = 0,1,2, \cdots$, of the differential equation (1.23).

CHAPTER IV

SOLUTIONS IN THE FINITE INTERVAL

The Finite Interval

The fundamental interval $[\alpha,\beta]$ may conceivably extend in either direction, or both directions, to infinity. These situations will be discussed in Chapters V and VI. This chapter will be devoted to the consideration of the finite interval $[\alpha,\beta]$, where $\alpha < \beta$.

The Weight Function $w(x)$

The satisfaction of the nine orthogonality conditions developed in Chapter II and stated in its summary depends on the choice of $w(x)$. Conversely, the choice of $w(x)$ is vital only with respect to the part it plays in the satisfaction of these orthogonality conditions.

A close scrutiny of the conditions will reveal a form of $w(x)$ which is sufficient to accomplish this aim.

From orthogonality condition (i) it is seen that the possibility of $w(x)$ vanishing at $x = \alpha$ and at $x = \beta$ must be included.

Since condition (ii) requires that $(wF)' = w'F + wF' = 0$ at $x = \alpha$ and $x = \beta$, the possibility that $w'(x)$ vanishes at $x = \alpha$ and at $x = \beta$ must also be included.

From orthogonality condition (iii), which states that $(wF)'' = w''F + 2w'F' + wF'' = 0$ at $x = \alpha$ and $x = \beta$, it is seen that w must be such that $w''(x)$ might vanish at $x = \alpha$ and $x = \beta$.

Orthogonality condition (iv) states that $wR - (wP)'' = 0$ at $x = \alpha$ and $x = \beta$, so this condition will be satisfied if $w(x)$, $w'(x)$, and $w''(x)$ vanish at $x = \alpha$ and $x = \beta$.

Condition (v) states that $(wR)' - 3(wP)''' = (wR)' - 3w'''P - 9w''P' - 9w'P'' - 3wP''' = 0$ at $x = \alpha$ and $x = \beta$, so the possibility that $w'''(x)$ vanishes at $x = \alpha$ and $x = \beta$ must be considered.

Condition (vi) states that $wT - (wR)'' + 3(wP)^{iv} = 0$ at $x = \alpha$ and $x = \beta$. Since $(wP)^{iv} = w^{iv}P + 4w'''P' + 6w''P'' + 4w'P''' + wP^{iv}$, the possibility that $w^{iv}(x)$ vanishes at $x = \alpha$ and $x = \beta$ must not be excluded.

Orthogonality conditions (vii), (viii), and (ix) are identities which state, respectively, that $wS = 2(w'R + wR') - 5(w'''P + 3w''P' + 3w'P'' + wP''')$, $wU = w'T + wT' - w'''R - 3w''R' - 3w'R'' - wR''' + 3(w^{v}P + 5w^{iv}P' + 10w'''P'' + 10w''P''' + 5w'P^{iv} + wP^{v})$, and $wQ = 3w'P + 3wP'$. Since S, U, and Q are polynomials, w must be a factor of the right member of each of these identities. This will be true if w', w'', w''', w^{iv}, and w^{v} can each be expressed as the product of w and a rational function in which the denominator divides the numerator.

Consider a choice of

$$(4.1) \quad w = (x-\alpha)^{g}(x-\beta)^{h},$$

where g and h are real numbers. Successive differentiations of equation (4.2) yield:

$$(4.2) \quad w' = g(x-\alpha)^{g-1}(x-\beta)^{h} + h(x-\alpha)^{g}(x-\beta)^{h-1}$$
$$= (x-\alpha)^{g}(x-\beta)^{h}[g(x-\beta) + h(x-\alpha)]/(x-\alpha)(x-\beta),$$

$$(4.3)\quad w'' = (g-1)g(x-\alpha)^{g-2}(x-\beta)^h + 2gh(x-\alpha)^{g-1}(x-\beta)^{h-1}$$
$$+ (h-1)h(x-\alpha)^g(x-\beta)^{h-2}$$
$$= (x-\alpha)^g(x-\beta)^h[(g-1)g(x-\beta)^2 + 2gh(x-\alpha)(x-\beta)$$
$$+ (h-1)h(x-\alpha)^2]/(x-\alpha)^2(x-\beta)^2,$$

$$(4.4)\quad w''' = (g-2)(g-1)g(x-\alpha)^{g-3}(x-\beta)^h + 3(g-1)gh(x-\alpha)^{g-2}(x-\beta)^{h-1}$$
$$+ 3g(h-1)h(x-\alpha)^{g-1}(x-\beta)^{h-2} + (h-2)(h-1)h(x-\alpha)^g(x-\beta)^{h-3}$$
$$= (x-\alpha)^g(x-\beta)^h[(g-2)(g-1)g(x-\beta)^3 + 3(g-1)gh(x-\alpha)(x-\beta)^2$$
$$+ 3g(h-1)h(x-\alpha)^2(x-\beta) + (h-2)(h-1)h(x-\alpha)^3]/(x-\alpha)^3(x-\beta)^3,$$

$$(4.5)\quad w^{iv} = (g-3)(g-2)(g-1)g(x-\alpha)^{g-4}(x-\beta)^h$$
$$+ 4(g-2)(g-1)gh(x-\alpha)^{g-3}(x-\beta)^{h-1}$$
$$+ 6(g-1)g(h-1)h(x-\alpha)^{g-2}(x-\beta)^{h-2}$$
$$+ 4g(h-2)(h-1)h(x-\alpha)^{g-1}(x-\beta)^{h-3}$$
$$+ (h-3)(h-2)(h-1)h(x-\alpha)^g(x-\beta)^{h-4}$$
$$= (x-\alpha)^g(x-\beta)^h[(g-3)(g-2)(g-1)g(x-\beta)^4$$
$$+ 4(g-2)(g-1)gh(x-\alpha)(x-\beta)^3$$
$$+ 6(g-1)g(h-1)h(x-\alpha)^2(x-\beta)^2 + 4g(h-2)(h-1)h(x-\alpha)^3(x-\beta)$$
$$+ (h-3)(h-2)(h-1)h(x-\alpha)^4]/(x-\alpha)^4(x-\beta)^4,$$

$$(4.6)\quad w^{v} = (g-4)(g-3)(g-2)(g-1)g(x-\alpha)^{g-5}(x-\beta)^h$$
$$+ 5(g-3)(g-2)(g-1)gh(x-\alpha)^{g-4}(x-\beta)^{h-1}$$
$$+ 10(g-2)(g-1)g(h-1)h(x-\alpha)^{g-3}(x-\beta)^{h-2}$$

$$+ 10(g-1)g(h-2)(h-1)h(x-\alpha)^{g-2}(x-\beta)^{h-3}$$

$$+ 5g(h-3)(h-2)(h-1)h(x-\alpha)^{g-1}(x-\beta)^{h-4}$$

$$+ (h-4)(h-3)(h-2)(h-1)h(x-\alpha)^{g}(x-\beta)^{h-5}$$

$$= (x-\alpha)^{g}(x-\beta)^{h}[(g-4)(g-3)(g-2)(g-1)g(x-\beta)^{5}$$

$$+ 5(g-3)(g-2)(g-1)gh(x-\alpha)(x-\beta)^{4}$$

$$+ 10\ (g-2)(g-1)g(h-1)h(x-\alpha)^{2}(x-\beta)^{3}$$

$$+ 10\ (g-1)g(h-2)(h-1)h(x-\alpha)^{3}(x-\beta)^{2}$$

$$+ 5g(h-3)(h-2)(h-1)h(x-\alpha)^{4}(x-\beta)$$

$$+ (h-4)(h-3)(h-2)(h-1)h(x-\alpha)^{5}]/(x-\alpha)^{5}(x-\beta)^{5}.$$

Thus, the choice of $w = (x-\alpha)^{g}(x-\beta)^{h}$, with suitable restrictions on g and h, will serve to satisfy the orthogonality conditions.

The Leading Coefficient $P(x)$

The determination of g and h in equation (4.1) is affected by whether or not $P(x) = 0$ has $x = \alpha$ and/or $x = \beta$ as solutions. A number of possibilities exist with regard to the roots of $P(x)$, and investigation of each of these possibilities will determine the form of the coefficients of the differential equation (1.23).

The assumption that $P(\alpha) \neq 0$. If $P(x)$ does not have $x = \alpha$ as a root, then the requirement of orthogonality condition (i) that $wP = (x-\alpha)^{g}(x-\beta)^{h}P = 0$ at $x = \alpha$ imposes the restriction

$$(4.7) \qquad g > 0,$$

since by assumption $P(\alpha) \neq 0$ and $\alpha < \beta$. But orthogonality condition (ix) requires

$$\begin{aligned}(x-\alpha)^g(x-\beta)^hQ &= 3[(x-\alpha)^g(x-\beta)^hP]' \\ &= 3[g(x-\alpha)^{g-1}(x-\beta)^hP + h(x-\alpha)^g(x-\beta)^{h-1}P \\ &\quad + (x-\alpha)^g(x-\beta)^hP'],\end{aligned}$$

or

$$(4.8)\quad Q = 3(x-\alpha)^{-1}(x-\beta)^{-1}[g(x-\beta)P + h(x-\alpha)P + (x-\alpha)(x-\beta)P'].$$

The polynomial nature of Q requires that $(x-\alpha)$ divide the expression in brackets in equation (4.8). But if this is true, then $g(\alpha-\beta)P(\alpha) = 0$, which is impossible since $\alpha < \beta$, $P(\alpha) \neq 0$, and by equation (4.7), $g > 0$. Hence the assumption that $P(\alpha) \neq 0$ is not valid.

<u>The assumption that $P(\beta) \neq 0$.</u> This assumption leads to a contradiction in a manner similar to that of the preceding section, as may be seen by interchanging the roles of α and β and of g and h in the proof of contradiction of the assumption in that section.

Consequently, $P(x)$ must have both $x = \alpha$ and $x = \beta$ as at least simple roots.

<u>The assumption that $P(x)$ has α as a simple root.</u> From the preceding section it was found that $P(x) = (x-\alpha)(x-\beta)V(x)$, and by the assumption in this section, $V(x)$ is a polynomial such that $V(\alpha) \neq 0$. Since $P(x)$ is a polynomial of degree six, at most, $V(x)$ must be of degree four, at most. Thus the fifth and all higher derivatives of $V(x)$ must vanish.

Orthogonality condition (i) requires that

(4.9) $wP = (x-\alpha)^{g+1}(x-\beta)^{h+1}V = 0$ at $x = \alpha$,

which implies

(4.10) $g > -1$

since $V(\alpha) \neq 0$ and $\alpha < \beta$.

Condition (ii) requires

(4.11)
$$(wP)' = (g+1)(x-\alpha)^{g}(x-\beta)^{h+1}V + (h+1)(x-\alpha)^{g+1}(x-\beta)^{h}V + (x-\alpha)^{g+1}(x-\beta)^{h+1}V'$$
$$= (x-\alpha)^{g}(x-\beta)^{h}[(g+1)(x-\beta)V + (h+1)(x-\alpha)V + (x-\alpha)(x-\beta)V'] = 0 \text{ at } x = \alpha.$$

If $g > 0$, condition (ii) is satisfied at $x = \alpha$. However if $-1 < g \leq 0$, then $(g+1)(\alpha-\beta)V(\alpha) = 0$, which is impossible since $V(\alpha) \neq 0$, $\alpha < \beta$. Hence

(4.12) $g > 0$.

Condition (iii) requires

(4.13)
$$(wP)'' = g(g+1)(x-\alpha)^{g-1}(x-\beta)^{h+1}V + 2(g+1)(h+1)(x-\alpha)^{g}(x-\beta)^{h}V + 2(g+1)(x-\alpha)^{g}(x-\beta)^{h+1}V' + h(h+1)(x-\alpha)^{g+1}(x-\beta)^{h-1}V + 2(h+1)(x-\alpha)^{g+1}(x-\beta)^{h}V' + (x-\alpha)^{g+1}(x-\beta)^{h+1}V''$$
$$= (x-\alpha)^{g-1}(x-\beta)^{h-1}[g(g+1)(x-\beta)^2V + 2(g+1)(h+1)(x-\alpha)(x-\beta)V + 2(g+1)(x-\alpha)(x-\beta)^2V' + h(h+1)(x-\alpha)^2V + 2(h+1)(x-\alpha)^2(x-\beta)V' + (x-\alpha)^2(x-\beta)^2V''] = 0 \text{ at } x = \alpha.$$

If $g > 1$, the condition is satisfied at $x = \alpha$. However if $0 < g \leq 1$, then $g(g+1)(\alpha-\beta)^2 V(\alpha) = 0$, which is impossible since $V(\alpha) \neq 0$, $\alpha < \beta$. Hence

$$g > 1. \tag{4.14}$$

Now consider condition (vii), which requires

$$\begin{aligned}
(x-\alpha)^g(x-\beta)^h S &= 2[(x-\alpha)^g(x-\beta)^h R]' - 5[(x-\alpha)^{g+1}(x-\beta)^{h+1}V]''' \\
&= 2g(x-\alpha)^{g-1}(x-\beta)^h R + 2h(x-\alpha)^g(x-\beta)^{h-1}R \\
&\quad + 2(x-\alpha)^g(x-\beta)^h R' - 5(g-1)g(g+1)(x-\alpha)^{g-2}(x-\beta)^{h+1}V \\
&\quad - 15g(g+1)(h+1)(x-\alpha)^{g-1}(x-\beta)^h V - 15(g+1)h(h+1)(x-\alpha)^g(x-\beta)^{h-1}V \\
&\quad - 5(h-1)h(h+1)(x-\alpha)^{g+1}(x-\beta)^{h-2}V - 30(g+1)(h+1)(x-\alpha)^g(x-\beta)^h V' \\
&\quad - 15g(g+1)(x-\alpha)^{g-1}(x-\beta)^{h+1}V' - 15h(h+1)(x-\alpha)^{g+1}(x-\beta)^{h-1}V' \\
&\quad - 15(h+1)(x-\alpha)^{g+1}(x-\beta)^h V'' - 15(g+1)(x-\alpha)^g(x-\beta)^{h+1}V'' \\
&\quad - 5(x-\alpha)^{g+1}(x-\beta)^{h+1}V''',
\end{aligned}$$

or

$$\begin{aligned}
S &= (x-\alpha)^{-2}(x-\beta)^{-2}[2g(x-\alpha)(x-\beta)^2 R + 2h(x-\alpha)^2(x-\beta)R \\
&\quad + 2(x-\alpha)^2(x-\beta)^2 R' - 5(g-1)g(g+1)(x-\beta)^3 V \\
&\quad - 15g(g+1)(h+1)(x-\alpha)(x-\beta)^2 V \\
&\quad - 15(g+1)h(h+1)(x-\alpha)^2(x-\beta)V - 5(h-1)h(h+1)(x-\alpha)^3 V \\
&\quad - 30(g+1)(h+1)(x-\alpha)^2(x-\beta)^2 V' - 15g(g+1)(x-\alpha)(x-\beta)^3 V' \\
&\quad - 15h(h+1)(x-\alpha)^3(x-\beta)V' - 15(h+1)(x-\alpha)^3(x-\beta)^2 V'' \\
&\quad - 15(g+1)(x-\alpha)^2(x-\beta)^3 V'' - 5(x-\alpha)^3(x-\beta)^3 V'''].
\end{aligned} \tag{4.15}$$

The polynomial nature of S requires that $(x-\alpha)$ divide the expression in brackets in equation (4.15). But this requires that $-5(g-1)g(g+1)(\alpha-\beta)^3V(\alpha) = 0$, which is impossible under the assumption $V(\alpha) \neq 0$, since $\alpha < \beta$ and, by equation (4.14), $g > 1$. Hence the assumption that $P(x)$ has α as a simple root is not valid.

<u>The assumption that $P(x)$ has β as a simple root.</u> This assumption is likewise invalid, as may be seen by interchanging the roles of α and β and of g and h in the proof of contradiction of the assumption in the preceding section.

<u>The assumption that $P(x)$ has α as a double root.</u> In this section it will be assumed that $P(x) = (x-\alpha)^2(x-\beta)^2V(x)$, where $V(x)$ is a polynomial such that $V(\alpha) \neq 0$. Since $P(x)$ is a polynomial of degree six, at most, $V(x)$ must be of degree two, at most, and hence the third and all higher derivatives of $V(x)$ must vanish.

Orthogonality condition (i) requires that

$$(4.16)\quad wP = (x-\alpha)^{g+2}(x-\beta)^{h+2}V = 0 \quad \text{at} \quad x = \alpha,$$

which implies

$$(4.17)\quad g > -2,$$

since $V(\alpha) \neq 0$ and $\alpha < \beta$.

Condition (ii) requires

$$(4.18)\quad (wP)' = (g+2)(x-\alpha)^{g+1}(x-\beta)^{h+2}V + (h+2)(x-\alpha)^{g+2}(x-\beta)^{h+1}V + (x-\alpha)^{g+2}(x-\beta)^{h+2}V'$$

$$= (x-\alpha)^{g+1}(x-\beta)^{h+1}[(g+2)(x-\beta)V + (h+2)(x-\alpha)V$$

$$+ (x-\alpha)(x-\beta)V'] = 0 \quad \text{at} \quad x = \alpha.$$

If $g > -1$, the condition is satisfied at $x = \alpha$. However if $-2 < g \leq -1$, then $(g+2)(\alpha-\beta)V(\alpha) = 0$, which is impossible since $V(\alpha) \neq 0$, $\alpha < \beta$. Hence

(4.19) $g > -1$.

Condition (iii) requires

$$(4.20) \quad (WP)'' = (g+1)(g+2)(x-\alpha)^g(x-\beta)^{h+2}V + 2(g+2)(h+2)(x-\alpha)^{g+1}(x-\beta)^{h+1}V$$

$$+ 2(g+2)(x-\alpha)^{g+1}(x-\beta)^{h+2}V' + (h+1)(h+2)(x-\alpha)^{g+2}(x-\beta)^hV$$

$$+ 2(h+2)(x-\alpha)^{g+2}(x-\beta)^{h+1}V' + (x-\alpha)^{g+2}(x-\beta)^{h+2}V''$$

$$= (x-\alpha)^g(x-\beta)^h[(g+1)(g+2)(x-\beta)^2V + 2(g+2)(h+2)(x-\alpha)(x-\beta)V$$

$$+ 2(g+2)(x-\alpha)(x-\beta)^2V' + (h+1)(h+2)(x-\alpha)^2V$$

$$+ 2(h+2)(x-\alpha)^2(x-\beta)V' + (x-\alpha)^2(x-\beta)^2V''] = 0 \quad \text{at} \quad x = \alpha.$$

If $g > 0$, the condition is satisfied at $x = \alpha$. However if $-1 < g \leq 0$, then $(g+1)(g+2)(\alpha-\beta)^2V(\alpha) = 0$, which is impossible since $V(\alpha) \neq 0$, $\alpha < \beta$. Hence

(4.21) $g > 0$.

Now consider condition (vii), which requires

$$(x-\alpha)^g(x-\beta)^hS = 2[(x-\alpha)^g(x-\beta)^hR]' - S[(x-\alpha)^{g+2}(x-\beta)^{h+2}V]'''$$

$$= 2g(x-\alpha)^{g-1}(x-\beta)^hR + 2h(x-\alpha)^g(x-\beta)^{h-1}R$$

$$+ 2(x-\alpha)^g(x-\beta)^h R' - 5g(g+1)(g+2)(x-\alpha)^{g-1}(x-\beta)^{h+2}V$$

$$- 15(g+1)(g+2)(h+2)(x-\alpha)^g(x-\beta)^{h+1}V$$

$$- 15(g+1)(g+2)(x-\alpha)^g(x-\beta)^{h+2}V'$$

$$- 30(g+2)(h+2)(x-\alpha)^{g+1}(x-\beta)^{h+1}V'$$

$$- 15(g+2)(h+1)(h+2)(x-\alpha)^{g+1}(x-\beta)^h V$$

$$- 5h(h+1)(h+2)(x-\alpha)^{g+2}(x-\beta)^{h-1}V$$

$$- 15(h+1)(h+2)(x-\alpha)^{g+2}(x-\beta)^h V'$$

$$- 15(h+2)(x-\alpha)^{g+2}(x-\beta)^{h+1}V''$$

$$- 15(g+2)(x-\alpha)^{g+1}(x-\beta)^{h+2}V'',$$

or

$$(4.22)\quad S = (x-\alpha)^{-1}(x-\beta)^{-1}[2g(x-\alpha)(x-\beta)^2R + 2h(x-\alpha)^2(x-\beta)R$$

$$+ 2(x-\alpha)^2(x-\beta)^2R'$$

$$- 5g(g+1)(g+2)(x-\beta)^3V - 15(g+1)(g+2)(h+2)(x-\alpha)(x-\beta)^2V$$

$$- 15(g+1)(g+2)(x-\alpha)(x-\beta)^3V' - 30(g+2)(h+2)(x-\alpha)^2(x-\beta)^2V'$$

$$- 15(g+2)(h+1)(h+2)(x-\alpha)^2(x-\beta)V - 5h(h+1)(h+2)(x-\alpha)^3V$$

$$- 15(h+1)(h+2)(x-\alpha)^3(x-\beta)V' - 15(h+2)(x-\alpha)^3(x-\beta)^2V''$$

$$- 15(g+2)(x-\alpha)^2(x-\beta)^3V''].$$

The polynomial nature of S requires that $(x-\alpha)$ divide the expression in brackets in equation (4.22). But this requires that $-5g(g+1)(g+2)(\alpha-\beta)^3V(\alpha) = 0$, which is impossible under the assumption $V(\alpha) \neq 0$, since $\alpha < \beta$ and, by equation (4.21), $g > 0$. Hence the assumption that $P(x)$ has α as a double root is not valid.

<u>The assumption that $P(x)$ has β as a double root.</u> This assumption is likewise invalid, as may be seen by interchanging the roles of α and β and of g and h in the proof of contradiction of the assumption in the preceding section.

<u>Verification that $P(x) = K(x-\alpha)^3(x-\beta)^3$, where K is a non-zero constant.</u> The contradictions of the assumptions made in the six preceding sections leave only one possibility: $P(x)$ has both α and β as triple roots. This must be the case since the coefficient $P(x)$ was determined in Chapter I to be a polynomial of degree six, at most, and in this chapter it was determined that $P(x)$ must have both $x = \alpha$ and $x = \beta$ as roots.

The nine orthogonality conditions will now be applied with $P = K(x-\alpha)^3(x-\beta)^3$, $w = (x-\alpha)^g(x-\beta)^h$.

Orthogonality condition (i) requires that

$$(4.23)\quad wP = K(x-\alpha)^{g+3}(x-\beta)^{h+3} = 0 \quad \text{at} \quad x = \alpha \quad \text{and at} \quad x = \beta,$$

which implies

$$(4.24)\quad g > -3$$

and

$$(4.25)\quad h > -3,$$

since $K \neq 0$ and $\alpha < \beta$.

Condition (ii) requires

$$(4.26)\quad (wP)' = K[(g+3)(x-\alpha)^{g+2}(x-\beta)^{h+3} + (h+3)(x-\alpha)^{g+3}(x-\beta)^{h+2}]$$
$$= K(x-\alpha)^{g+2}(x-\beta)^{h+2}[(g+3)(x-\beta) + (h+3)(x-\alpha)] = 0 \quad \text{at} \quad x = \alpha \quad \text{and} \quad x = \beta.$$

If $g > -2$, the condition is satisfied at $x = \alpha$. However if $-3 < g \leq -2$, then $(g+3)(\alpha-\beta) = 0$, which is impossible since $\alpha < \beta$. Hence

$$(4.27)\quad g > -2.$$

Similarly, if $h > -2$, the condition is satisfied at $x = \beta$. However if $-3 < h \leq -2$, then $(h+3)(\beta-\alpha) = 0$, which is impossible since $\alpha < \beta$. Hence

$$(4.28)\quad h > -2.$$

Now application of condition (ii) yields [see equation (4.26)]
$$(x-\alpha)^g(x-\beta)^h Q = 3K(x-\alpha)^{g+2}(x-\beta)^{h+2}[(g+3)(x-\beta) + (h+3)(x-\alpha)],$$

or

$$(4.29)\quad Q = 3K(x-\alpha)^2(x-\beta)^2[(g+3)(x-\beta) + (h+3)(x-\alpha)].$$

Condition (iii) requires

$$(4.30)\quad (wP)'' = K[(g+2)(g+3)(x-\alpha)^{g+1}(x-\beta)^{h+3} + 2(g+3)(h+3)(x-\alpha)^{g+2}(x-\beta)^{h+2} + (h+2)(h+3)(x-\alpha)^{g+3}(x-\beta)^{h+1}]$$
$$= K(x-\alpha)^{g+1}(x-\beta)^{h+1}[(g+2)(g+3)(x-\beta)^2 + 2(g+3)(h+3)(x-\alpha)(x-\beta) + (h+2)(h+3)(x-\alpha)^2] = 0$$
$$\text{at} \quad x = \alpha \quad \text{and} \quad x = \beta.$$

If $g > -1$, the condition is satisfied at $x = \alpha$. However if $-2 < g \leq -1$, then $(g+2)(g+3)(\alpha-\beta)^2 = 0$, which is impossible since $\alpha < \beta$. Hence

$$(4.31) \quad g > -1.$$

Similarly,

$$(4.32) \quad h > -1.$$

Condition (iv) requires

$$\begin{aligned}(4.33) \quad wR - (W^2)'' &= (x-\alpha)^g(x-\beta)^h R - K(x-\alpha)^{g+1}(x-\beta)^{h+1}[(g+2)(g+3)(x-\beta)^2 \\ &\quad + 2(g+3)(h+3)(x-\alpha)(x-\beta) + (h+2)(h+3)(x-\alpha)^2] \\ &= (x-\alpha)^g(x-\beta)^h \{R - K(x-\alpha)(x-\beta)[(g+2)(g+3)(x-\beta)^2 \\ &\quad + 2(g+3)(h+3)(x-\alpha)(x-\beta) + (h+2)(h+3)(x-\alpha)^2]\} \\ &= 0 \quad \text{at} \quad x = \alpha \quad \text{and} \quad x = \beta.\end{aligned}$$

If $g > 0$, the condition is satisfied at $x = \alpha$; if $h > 0$, the condition is satisfied at $x = \beta$. If $-1 < g \leq 0$, then the condition is satisfied at $x = \alpha$ only if

$$(4.34) \quad R(\alpha) = 0, \quad \text{for} \quad -1 < g \leq 0.$$

Similarly, the condition is satisfied at $x = \beta$ only if

$$(4.35) \quad R(\beta) = 0, \quad \text{for} \quad -1 < h \leq 0.$$

Condition (v) requires

$$
\begin{aligned}
(4.36)\quad (wR)' - 3(wP)''' &= (x-\alpha)^{g-1}(x-\beta)^{h-1}[g(x-\beta)R + h(x-\alpha)R \\
&+ (x-\alpha)(x-\beta)R'] - 3K[(g+1)(g+2)(g+3)(x-\alpha)^{g}(x-\beta)^{h+3} \\
&+ 3(g+2)(g+3)(h+3)(x-\alpha)^{g+1}(x-\beta)^{h+2} \\
&+ 3(g+3)(h+2)(h+3)(x-\alpha)^{g+2}(x-\beta)^{h+1} \\
&+ (h+1)(h+2)(h+3)(x-\alpha)^{g+3}(x-\beta)^{h}] \\
&= (x-\alpha)^{g-1}(x-\beta)^{h-1}\Big[g(x-\beta)R + h(x-\alpha)R \\
&+ (x-\alpha)(x-\beta)\{R' - 3K[(g+1)(g+2)(g+3)(x-\beta)^{3} \\
&+ 3(g+2)(g+3)(h+3)(x-\alpha)(x-\beta)^{2} \\
&+ 3(g+3)(h+2)(h+3)(x-\alpha)^{2}(x-\beta) \\
&+ (h+1)(h+2)(h+3)(x-\alpha)^{3}]\}\Big] = 0 \text{ at } x = \alpha \text{ and } x = \beta.
\end{aligned}
$$

If $g > 1$, the condition is satisfied at $x = \alpha$; if $h > 1$, the condition is satisfied at $x = \beta$. If $0 < g \leq 1$, then the condition is satisfied at $x = \alpha$ only if $g(\alpha-\beta)R(\alpha) = 0$, or, since $\alpha < \beta$,

$$(4.37)\quad R(\alpha) = 0, \quad \text{for} \quad 0 < g \leq 1.$$

Similarly, the condition is satisfied at $x = \beta$ only if

$$(4.38)\quad R(\beta) = 0, \quad \text{for} \quad 0 < h \leq 1.$$

If $-1 < g \leq 0$, then by equation (4.34), $R(\alpha) = 0$, and hence condition (v) is satisfied at $x = \alpha$ only if in addition,

$$(4.39)\quad R'(\alpha) = 3K(g+1)(g+2)(g+3)(\alpha-\beta)^{3}, \quad \text{for} \quad -1 < g \leq 0.$$

Similarly, condition (v) is satisfied at $x = \beta$ only if equation (4.35) holds and

$$(4.40)\quad R'(\beta) = 3K(h+1)(h+2)(h+3)(\beta-\alpha)^5, \qquad \text{for } -1 < h \leq 0.$$

Condition (vi) requires

$$
\begin{aligned}
(4.41)\quad wT - (wR)'' + 3(wP)^{iv} = {} & (x-\alpha)^g(x-\beta)^hT - (g-1)g(x-\alpha)^{g-2}(x-\beta)^hR \\
& - (h-1)h(x-\alpha)^g(x-\beta)^{h-2}R - 2gh(x-\alpha)^{g-1}(x-\beta)^{h-1}R \\
& - (x-\alpha)^g(x-\beta)^hR'' - 2h(x-\alpha)^g(x-\beta)^{h-1}R' - 2g(x-\alpha)^{g-1}(x-\beta)^hR' \\
& + 3K[g(g+1)(g+2)(g+3)(x-\alpha)^{g-1}(x-\beta)^{h+3} \\
& + 4(g+1)(g+2)(g+3)(h+3)(x-\alpha)^g(x-\beta)^{h+2} \\
& + g(g+2)(g+3)(h+2)(h+3)(x-\alpha)^{g+1}(x-\beta)^{h+2} \\
& + 4(g+3)(h+1)(h+2)(h+3)(x-\alpha)^{g+2}(x-\beta)^{h+2} \\
& + h(h+1)(h+2)(h+3)(x-\alpha)^{g+3}(x-\beta)^{h-1}] \\
& \qquad = (x-\alpha)^{g-2}(x-\beta)^{h-2}\Big[-(g-1)g(x-\beta)^2R \\
& - (h-1)h(x-\alpha)^2R + (x-\alpha)(x-\beta)\{-2ghR - 2h(x-\alpha)R' \\
& - 2g(x-\beta)R' + 3Kg(g+1)(g+2)(g+3)(x-\beta)^4 \\
& + 3Kh(h+1)(h+2)(h+3)(x-\alpha)^4 \\
& + (x-\alpha)(x-\beta)[12K(g+1)(g+2)(g+3)(h+3)(x-\beta)^2 \\
& + 12K(g+3)(h+1)(h+2)(h+3)(x-\alpha)^2 + T - R''
\end{aligned}
$$

$$+ 18K(g+2)(g+3)(h+2)(h+3)(x-\alpha)(x-\beta)]\}$$

$$= 0 \quad \text{at} \quad x = \alpha \quad \text{and} \quad x = \beta.$$

If $g > 2$, the condition is satisfied at $x = \alpha$; if $h > 2$, the condition is satisfied at $x = \beta$ If $1 < g \leq 2$, then the condition is satisfied at $x = \alpha$ only if $-(g-1)g(\alpha-\beta)^2R(\alpha) = 0$, or, since $\alpha < \beta$,

(4.42) $R(\alpha) = 0, \quad \text{for} \quad 1 < g \leq 2.$

Similarly, the condition is satisfied at $x = \beta$ only if

(4.43) $R(\beta) = 0, \quad \text{for} \quad 1 < h \leq 2.$

If $0 < g \leq 1$, then by equation (4.37), $R(\alpha) = 0$, and hence condition (vi) is satisfied at $x = \alpha$ only if in addition

(4.44) $-2g(\alpha-\beta)R'(\alpha) + 3Kg(g+1)(g+2)(g+3)(\alpha-\beta)^4 = 0 \quad \text{for} \quad 0 < g \leq 1,$

or, since $\alpha < \beta$,

(4.45) $R'(\alpha) = \frac{3K}{2}(g+1)(g+2)(g+3)(\alpha-\beta)^3, \quad \text{for } 0 < g \leq 1.$

Similarly, condition (vi) is satisfied at $x = \beta$ only if equation (4.38) holds and

(4.46) $R'(\beta) = \frac{3K}{2}(h+1)(h+2)(h+3)(\beta-\alpha)^3, \quad \text{for} \quad 0 < h \leq 1.$

If $-1 < g \leq 0$, then by equation (4.34), $R(\alpha) = 0$, and hence condition (vi) is satisfied at $x = \alpha$ only if in addition equation (4.44) holds for $-1 < g \leq 0$ as well as for $0 < g \leq 1$, and

(4.47) $T(\alpha) = R''(\alpha) - 12K(g+1)(g+2)(g+3)(h+3)(\alpha-\beta)^2, \quad \text{for} \quad -1 < g \leq 0.$

Since $\alpha < \beta$, equation (4.44) may be written for $-1 < g \leq 0$ as

(4.48) $gR'(\alpha) = \frac{3K}{2} g(g+1)(g+2)(g+3)(\alpha-\beta)^3$, for $-1 < g \leq 0$.

Similarly, condition (vi) is satisfied at $x = \beta$ only if equation (4.35) holds and

(4.49) $T(\beta) = R''(\beta) - 12K(g+3)(h+1)(h+2)(h+3)(\beta-\alpha)^2$ for $-1 < h \leq 0$,

(4.50) $hR'(\beta) = \frac{3K}{2} h(h+1)(h+2)(h+3)(\beta-\alpha)^3$ for $-1 < h \leq 0$.

Now equations (4.39) and (4.48) together imply $K = K/2$ for $-1 < g < 0$. But this is impossible, since $K \neq 0$, so the orthogonality conditions cannot be satisfied for $-1 < g < 0$. Thus, in view of equation (4.31),

(4.51) $g \geq 0$.

Similarly, equations (4.40), (4.50), and (4.32) require

(4.52) $h \geq 0$.

Now equations (4.34), (4.37), and (4.42) together and equations (4.35), (4.38), and (4.43) together require, respectively,

(4.53) $R(\alpha) = 0$, for $0 \leq g \leq 2$,

(4.54) $R(\beta) = 0$, for $0 \leq h \leq 2$.

For $g = 0$, equation (4.39) becomes

(4.55) $R'(\alpha) = 18K(\alpha-\beta)^3$, for $g = 0$.

Similarly, equation (4.40) becomes

$$(4.56)\quad R'(\beta) = 18K(\beta-\alpha)^3 \quad \text{for } h = 0;$$

equation (4.47) becomes

$$(4.57)\quad T(\alpha) = R''(\alpha) - 72K(h+3)(\alpha-\beta)^2 \quad \text{for } g = 0;$$

and equation (4.48) becomes

$$(4.58)\quad T(\beta) = R''(\beta) - 72K(g+3)(\beta-\alpha)^2 \quad \text{for } h = 0.$$

Orthogonality condition (vii) requires

$$(x-\alpha)^g(x-\beta)^h S = 2[(x-\alpha)^g(x-\beta)^h R]' - 5[K(x-\alpha)^{g+3}(x-\beta)^{h+3}]'''$$
$$= (x-\alpha)^{g-1}(x-\beta)^{h-1}\Big[2g(x-\beta)R + 2h(x-\alpha)R$$
$$+ (x-\alpha)(x-\beta)\{2R' - 5K[(g+1)(g+2)(g+3)(x-\beta)^3$$
$$+ 3(g+2)(g+3)(h+3)(x-\alpha)(x-\beta)^2$$
$$+ 3(g+3)(h+2)(h+3)(x-\alpha)^2(x-\beta)$$
$$+ (h+1)(h+2)(h+3)(x-\alpha)^3]\}\Big] \qquad \text{[See equation (4.38)]},$$

or

$$(4.59)\quad S = (x-\alpha)^{-1}(x-\beta)^{-1}\Big[2g(x-\beta)R + 2h(x-\alpha)R + (x-\alpha)(x-\beta)\{2R'$$
$$- 5K[(g+1)(g+2)(g+3)(x-\beta)^3 + 3(g+2)(g+3)(h+3)(x-\alpha)(x-\beta)^2$$
$$+ 3(g+3)(h+2)(h+3)(x-\alpha)^2(x-\beta) + (h+1)(h+2)(h+3)(x-\alpha)^3]\}\Big].$$

The polynomial nature of S requires that $(x-\alpha)(x-\beta)$ divide the expression in large brackets in the right member of equation (4.59).

If $(x-\alpha)$ divides this expression, then $2g(\alpha-\beta)R(\alpha) = 0$, or, since $\alpha < \beta$, $gR(\alpha) = 0$. This expression requires $R(\alpha) = 0$ for $g \neq 0$. But by equation (4.53), $R(\alpha) = 0$ for $g = 0$. Hence

(4.60) $R(\alpha) = 0.$

In a similar manner it may be seen that

(4.61) $R(\beta) = 0.$

Equation (4.58) for S may be written

(4.62) $$S = \frac{2gR}{x-\alpha} + \frac{2hR}{x-\beta} + 2R' - 5K[(g+1)(g+2)(g+3)(x-\beta)^3$$

$$+ 3(g+2)(g+3)(h+3)(x-\alpha)(x-\beta)^2$$

$$+ 3(g+3)(h+2)(h+3)(x-\alpha)^2(x-\beta)$$

$$+ (h+1)(h+2)(h+3)(x-\alpha)^3],$$

and since R is a polynomial of degree four, at most, equation (4.62) is a polynomial of degree three, at most.

Orthogonality condition (viii) requires

$$(x-\alpha)^g(x-\beta)^hU = [(x-\alpha)^g(x-\beta)^hT]' - [(x-\alpha)^g(x-\beta)^hR]'''$$

$$+ 3[K(x-\alpha)^{g+3}(x-\beta)^{h+3}]^v$$

$$= g(x-\alpha)^{g-1}(x-\beta)^hT + h(x-\alpha)^g(x-\beta)^{h-1}T$$

$$+ (x-\alpha)^g(x-\beta)^hT'$$

$$- (g-2)(g-1)g(x-\alpha)^{g-3}(x-\beta)^hR - (h-2)(h-1)h(x-\alpha)^g(x-\beta)^{h-3}R$$

$$- 3g(h-1)h(x-\alpha)^{g-1}(x-\beta)^{h-2}R - 3(g-1)gh(x-\alpha)^{g-2}(x-\beta)^{h-1}R$$

$$- (x-\alpha)^{g}(x-\beta)^{h}R''' - 3h(x-\alpha)^{g}(x-\beta)^{h-1}R''$$

$$- 3(h-1)h(x-\alpha)^{g}(x-\beta)^{h-2}R' - 3g(x-\alpha)^{g-1}(x-\beta)^{h}R''$$

$$- 6gh(x-\alpha)^{g-1}(x-\beta)^{h-1}R' - 3(g-1)g(x-\alpha)^{g-2}(x-\beta)^{h}R'$$

$$+ 3K[(g-1)g(g+1)(g+2)(g+3)(x-\alpha)^{g-2}(x-\beta)^{h+3}$$

$$+ 5g(g+1)(g+2)(g+3)(h+3)(x-\alpha)^{g-1}(x-\beta)^{h+2}$$

$$+ 10(g+1)(g+2)(g+3)(h+2)(h+3)(x-\alpha)^{g}(x-\beta)^{h+1}$$

$$+ 10(g+2)(g+3)(h+1)(h+2)(h+3)(x-\alpha)^{g+1}(x-\beta)^{h}$$

$$+ 5(g+3)h(h+1)(h+2)(h+3)(x-\alpha)^{g+2}(x-\beta)^{h-1}$$

$$+ (h-1)h(h+1)(h+2)(h+3)(x-\alpha)^{g+3}(x-\beta)^{h-2}],$$

or

$$(4.63) \qquad U = [gT - 3gR'' + 15Kg(g+1)(g+2)(g+3)(h+3)(x-\beta)^{2}]/(x-\alpha)$$

$$+ [hT - 3hR'' + 15K(g+3)h(h+1)(h+2)(h+3)(x-\alpha)^{2}]/(x-\beta)$$

$$- [3(g-1)gR' - 3K(g-1)g(g+1)(g+2)(g+3)(x-\beta)^{3}]/(x-\alpha)^{2}$$

$$- [3(h-1)hR' - 3K(h-1)h(h+1)(h+2)(h+3)(x-\alpha)^{3}]/(x-\beta)^{2}$$

$$- (g-2)(g-1)gR/(x-\alpha)^{3} - (h-2)(h-1)hR/(x-\beta)^{3}$$

$$- 6ghR'/(x-\alpha)(x-\beta) - 3g(h-1)hR/(x-\alpha)(x-\beta)^{2}$$

$$- 3(g-1)ghR/(x-\alpha)^{2}(x-\beta) + T' - R'''$$

$$+ 30K(g+1)(g+2)(g+3)(h+2)(h+3)(x-\beta)$$

$$+ 30K(g+2)(g+3)(h+1)(h+2)(h+3)(x-\alpha).$$

Since R and T are polynomials of degree at most four and two, respectively, equation (4.63) is of first degree. For convenience, let the numerators of the fractional terms in the right member of this equation be designated as follows

$$(4.64) \quad \gamma_1(x) = [gT - 3gR'' + 15Kg(g+1)(g+2)(g+3)(h+3)(x-\beta)^2],$$

$$(4.65) \quad \delta_1(x) = [hT - 3hR'' + 15K(g+3)h(h+1)(h+2)(h+3)(x-\alpha)^2],$$

$$(4.66) \quad \gamma_2(x) = [3(g-1)gR' - 5K(g-1)g(g+1)(g+2)(g+3)(x-\beta)^3],$$

$$(4.67) \quad \delta_2(x) = [3(h-1)hR' - 5K(h-1)h(h+1)(h+2)(h+3)(x-\alpha)^3],$$

$$(4.68) \quad N_g = (g-2)(g-1)g,$$

$$(4.69) \quad N_h = (h-2)(h-1)h,$$

$$(4.70) \quad N_1 = 6gh,$$

$$(4.71) \quad N_2 = 3g(h-1)h,$$

$$(4.72) \quad N_3 = 3(g-1)gh$$

The fractional terms in the right member of equation (4.63) may be combined to yield the single term

$$(4.73) \quad [(x-\alpha)^2(x-\beta)^3\gamma_1 + (x-\alpha)^3(x-\beta)^2\delta_1 - (x-\alpha)(x-\beta)^3\gamma_2$$

$$- (x-\alpha)^3(x-\beta)\delta_2 - (x-\beta)^3 N_g R - (x-\alpha)^3 N_h R - (x-\alpha)^2(x-\beta)^2 N_1 R'$$

$$- (x-\alpha)^2(x-\beta)N_2R - (x-\alpha)(x-\beta)^2N_3R]/(x-\alpha)^3(x-\beta)^3.$$

The polynomial nature of U requires that the numerator of the term represented by expression (4.73) be divisible by the denominator $(x-\alpha)^3(x-\beta)^3$ of the term. A necessary and sufficient condition that this be true is that the numerator and the first and second derivatives of the numerator vanish at $x = \alpha$ and $x = \beta$. If the numerator of expression (4.73) vanishes at $x = \alpha$, then $N_2R(\alpha) = 0$. But equation (4.60) has already imposed the condition $R(\alpha) = 0$, so no new restriction is obtained. Equation (4.61) requires $R(\beta) = 0$, so likewise no new restriction is obtained upon setting the numerator of term (4.73) equal to zero at $x = \beta$. If the first derivative of the numerator of (4.73) vanishes at $x = \alpha$, then $-(\alpha-\beta)^3v_2(\alpha) - (\alpha-\beta)^3N_2R'(\alpha) = 0$, or since $\alpha < \beta$,

$$(4.74) \qquad N_2R'(\alpha) = - v_2(\alpha).$$

Similarly, if the first derivative of the numerator of (4.73) vanishes at $x = \beta$, it follows that

$$(4.75) \qquad N_2R'(\beta) = - s_2(\beta).$$

If the second derivative of the numerator of (4.73) vanishes at $x = \alpha$,

$$2(\alpha-\beta)^3v_1(\alpha) - 3(\alpha-\beta)^2v_2(\alpha) + (\alpha-\beta)^3v_2'(\alpha) - 3(\alpha-\beta)^2v_2(\alpha)$$

$$- (\alpha-\beta)^3v_2'(\alpha) - 3(\alpha-\beta)^2N_2R'(\alpha) - 3(\alpha-\beta)^2N_2R'(\alpha)$$

$$- (\alpha-\beta)^3N_2R''(\alpha) - 2(\alpha-\beta)^2N_1R'(\alpha) - (\alpha-\beta)^2N_3R'(\alpha)$$

$$- (\alpha-\beta)^2N_3R'(\alpha) = 0,$$

or since $\alpha < \beta$,

$$(4.76)\quad 2(\alpha-\beta)\gamma_1(\alpha) - 8\gamma_2(\alpha) - 2(\alpha-\beta)\gamma_2'(\alpha) - 8N_gR'(\alpha) - (\alpha-\beta)N_gR''(\alpha)$$

$$- 2N_1R'(\alpha) - 2N_3R'(\alpha) = 0.$$

If the second derivative of the numerator of (4.75) vanishes at $x = \beta$, it follows in a similar manner that

$$(4.77)\quad 2(\beta-\alpha)\delta_1(\beta) - 8\delta_2(\beta) - 2(\beta-\alpha)\delta_2'(\beta) - 8N_hR'(\beta) - (\beta-\alpha)N_hR''(\beta)$$

$$- 2N_1R'(\beta) - 2N_2R'(\beta) = 0.$$

Substitution of the expressions for N_g, $\gamma_2(\alpha)$, N_h, and $\delta_2(\beta)$ from equations (4.68), (4.69), (4.86), and (4.87) into equations (4.74) and (4.75) yields, after division by $(g+1)$ and $(h+1)$ respectively [by equations (4.51) and (4.52), $(g+1) > 0$ and $(h+1) > 0$],

$$(4.78)\quad (g-2)gR'(\alpha) = 3K(g-1)g(g+2)(g+3)(\alpha-\beta)^3,$$

$$(4.79)\quad (h-1)hR'(\beta) = 3K(h-1)h(h+2)(h+3)(\beta-\alpha)^3.$$

Equations (4.78) and (4.79) may be written

$$(4.80)\qquad R'(\alpha) = 3K(g+2)(g+3)(\alpha-\beta)^3 \quad \text{for} \quad g \neq 0, \quad g \neq 1,$$

$$(4.81)\qquad R'(\beta) = 3K(h+2)(h+3)(\beta-\alpha)^3 \quad \text{for} \quad h \neq 0, \quad h \neq 1.$$

Substitution of the expressions for $\gamma_1(\alpha)$, $\gamma_2(\alpha)$, N_g, N_h, N_1, N_2, N_3, $\delta_1(\beta)$, and $\delta_2(\beta)$ from equations (4.64), (4.65), (4.68), (4.69), (4.70), (4.71), (4.72), (4.86), and (4.87) into equations (4.76) and (4.77) yields

$$(4.82)\quad 2g(\alpha-\beta)T(\alpha) = g(g+1)(g+2)(\alpha-\beta)R''(\alpha) + 6g(g+1)(g+h-1)R'(\alpha)$$

$$-6Kg(g+1)(g+2)(g+3)(6g+5h+9)(\alpha-\beta)^3,$$

$$(4.83)\quad 2h(\beta-\alpha)T(\beta) = h(h+1)(h+2)(\beta-\alpha)R''(\beta) + 6h(h+1)(g+h-1)R'(\beta)$$

$$-6Kh(h+1)(h+2)(h+3)(5g+6h+9)(\beta-\alpha)^3.$$

Equations (4.82) and (4.83) may be written

$$(4.84)\quad T(\alpha) = \frac{1}{2}(g+1)(g+2)R''(\alpha) + \frac{3}{\alpha-\beta}(g+1)(g+h-1)R'(\alpha)$$

$$- 3K(g+1)(g+2)(g+3)(6g+5h+9)(\alpha-\beta)^2 \quad \text{for } g \neq 0,$$

$$(4.85)\quad T(\beta) = \frac{1}{2}(h+1)(h+2)R''(\beta) + \frac{3}{\beta-\alpha}(h+1)(g+h-1)R'(\beta)$$

$$- 3K(h+1)(h+2)(h+3)(5g+6h+9)(\beta-\alpha)^2 \quad \text{for } h \neq 0.$$

Now equations (4.45) and (4.80) together imply $\frac{K}{2}(g+1) = K$ for $0 < g < 1$. But this is impossible, since $K \neq 0$, $g \geq 0$. Thus the orthogonality conditions cannot be satisfied for $0 < g < 1$, so in view of equation (4.31),

$$(4.86)\quad g = 0 \text{ or } g \geq 1.$$

Similarly, equations (4.46), (4.81), and (4.32) require

$$(4.87)\quad h = 0 \text{ or } h \geq 1.$$

For $g = 1$, equation (4.45) becomes

$$(4.88)\quad R'(\alpha) = 3\kappa K(\alpha-\beta)^3, \quad \text{for } g = 1.$$

Similarly, equation (4.46) becomes

$$(4.89) \qquad R'(\beta) = 36K(\beta-\alpha)^3, \quad \text{for } h = 1.$$

Careful comparison of equation (4.80) with equations (4.55) and (4.88) reveals

$$(4.90) \qquad R'(\alpha) = 3K(g+2)(g+3)(\alpha-\beta)^3.$$

Similar comparison of equations (4.81), (4.58), and (4.88) yields

$$(4.91) \qquad R'(\beta) = 3K(h+2)(h+3)(\beta-\alpha)^3.$$

Equations (4.28), (4.57), (4.58), (4.80) through (4.83), (4.84) through (4.87), (4.90), and (4.91) constitute the totality of restrictions imposed on coefficients Q, R, S, T, and U by the nine orthogonality conditions with $P = K(x-\alpha)^3(x-\beta)^3$, $K \neq 0$, and $w = (x-\alpha)^g(x-\beta)^h$.

Summary

The choice of $w(x) = (x-\alpha)^g(x-\beta)^h$, where $g = 0$ or $g \geq 1$ and $h = 0$ or $h \geq 1$, will serve as a weight function in the orthogonalization of the solution set $\{y_n\}$, $n = 0,1,2, \ldots$, represented by equation (1.2) of the differential equation (1.23) over the fundamental interval $[\alpha,\beta]$ where both α and β are finite.

This choice of $w(x)$, however, imposes certain restrictions on the coefficients P, Q, R, S, T, and U of the differential equation (1.23) in the satisfaction of the nine orthogonality conditions. It was found that the coefficient P of y_n^{vi} must be of the form

$$P = K(x-\alpha)^3(x-\beta)^3, \quad K \neq 0,$$

where K is a constant. With this form of P it was found that the coefficient Q of y_n^{v} can be written

$$Q = 3K(x-\alpha)^2(x-\beta)^2[(g+3)(x-\beta) + (h+3)(x-\alpha)].$$

The coefficient R of y_n^{iv} is of the form $R = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$, A_i = constant, $i = 0,1,2,3,4$, and it was found that

$$R(\alpha) = R(\beta) = 0,$$

$$R'(\alpha) = 3K(g+2)(g+3)(\alpha-\beta)^3,$$

$$R'(\beta) = 3K(h+2)(h+3)(\beta-\alpha)^3.$$

The coefficient S of y_n''' was found to be defined by

$$S = \frac{2gR}{x-\alpha} + \frac{2hR}{x-\beta} + 2R' - 5K[(g+1)(g+2)(g+3)(x-\beta)^3$$

$$+ 3(g+2)(g+3)(h+3)(x-\alpha)(x-\beta)^2 + 3(g+3)(h+2)(h+3)(x-\alpha)^2(x-\beta)$$

$$+ (h+1)(h+2)(h+3)(x-\alpha)^3].$$

The coefficient T of y_n'' is of the form $T = B_0x^2 + B_1x + B_2$, B_i = constant, $i = 0,1,2$, and it was found that

$$T(\alpha) = R''(\alpha) - 72K(h+3)(\alpha-\beta)^2 \quad \text{for } g = 0,$$

$$T(\beta) = R''(\beta) - 72K(g+3)(\beta-\alpha)^2 \quad \text{for } h = 0,$$

$$T(\alpha) = \frac{1}{2}(g+1)(g+2)R''(\alpha) + \frac{3}{\alpha-\beta}(g+1)(g+h-1)R'(\alpha)$$

$$- 3K(g+1)(g+2)(g+3)(5g+3h+3)(\alpha-\beta)^2 \quad \text{for } g \neq 0,$$

$$T(\beta) = \frac{1}{2}(h+1)(h+2)R''(\beta) + \frac{3}{\beta-\alpha}(h+1)(g+h-1)R'(\beta)$$

$$- 3K(h+1)(h+2)(h+3)(5g+8h+5)(\beta-\alpha)^2 \quad \text{for} \quad h \neq 0.$$

The coefficient U of y_n' was found to be defined by

$$U = [gT - 3gR'' + 15Kg(g+1)(g+2)(g+3)(h+3)(x-\beta)^2]/(x-\alpha)$$

$$+ [hT - 3hR'' + 15K(g+3)h(h+1)(h+2)(h+3)(x-\alpha)^2]/(x-\beta)$$

$$- [3(g-1)gR' - 3K(g-1)g(g+1)(g+2)(g+3)(x-\beta)^3]/(x-\alpha)^2$$

$$- [3(h-1)hR' - 3K(h-1)h(h+1)(h+2)(h+3)(x-\alpha)^3]/(x-\beta)^2$$

$$- (g-2)(g-1)gR/(x-\alpha)^3 - (h-2)(h-1)hR/(x-\beta)^3$$

$$- 6ghR'/(x-\alpha)(x-\beta) - 3g(h-1)hR/(x-\alpha)(x-\beta)^2$$

$$- 3(g-1)ghR/(x-\alpha)^2(x-\beta) + T' - R'''$$

$$+ 30K(g+1)(g+2)(g+3)(h+2)(h+3)(x-\beta)$$

$$+ 30K(g+2)(g+3)(h+1)(h+2)(h+3)(x-\alpha).$$

Example of the Finite Interval

Choose $g = h = 0$, $\alpha = -1$ and $\beta = 1$. Then

(4.E1) $\quad W = 1.$

(4.E2) $\quad P = K(x+1)^3(x-1)^3 = K(x^6-3x^4+3x^2-1).$

(4.E3) $\quad Q = 3K(x+1)^2(x-1)^2[3(x-1) + 3(x+1)] = 18K(x^5-2x^3+x)$

(4.E4) $\quad R = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4,$

where

(4.E5) $\quad R(-1) = R(1) = 0,$

(4.E6) $\quad R'(-1) = 3K(2)(3)(-1-1)^3 = -144K,$

(4.E7) $\quad R'(1) = 3K(2)(3)(1+1)^3 = 144K.$

Successive differentiation of (4.E4) yields

(4.E8) $\quad R' = 4A_0x^3 + 3A_1x^2 + 2A_2x + A_3,$

(4.E9) $\quad R'' = 12A_0x^2 + 6A_1x + 2A_2,$

(4.E10) $\quad R''' = 24A_0x + 6A_1.$

From equations (4.E4) and (4.E5),

(4.E11) $\quad R(1) = A_0 + A_1 + A_2 + A_3 + A_4 = 0,$

(4.E12) $\quad R(-1) = A_0 - A_1 + A_2 - A_3 + A_4 = 0.$

Solution of equations (4.E11) and (4.E12) yields

(4.E13) $$A_4 = -A_0 - A_2,$$

(4.E14) $$A_3 = -A_1$$

From equations (4.E5), (4.E6), and (4.E7),

(4.E15) $$R'(1) = 4A_0 + 3A_1 + 2A_2 + A_3 = 144K,$$

(4.E16) $$R'(-1) = -4A_0 + 3A_1 - 2A_2 + A_3 = -144K.$$

Solution of equations (4.E15) and (4.E16) yields

(4.E17) $$A_2 = 72K - 2A_0,$$

(4.E18) $$A_3 = -3A_1$$

Equations (4.E14) and (4.E18) together imply

(4.E19) $$A_1 = A_3 = 0$$

Substitution of equations (4.E13), (4.E17), and (4.E19) into equations (4.E4), (4.E8), (4.E9), and (4.E10) yields

(4.E20) $$R = A_0x^4 + (72K-2A_0)x^2 + (A_0-72K),$$

(4.E21) $$R' = 4A_0x^3 + (144K-4A_0)x,$$

(4.E22) $$R'' = 12A_0x^2 + (144K-4A_0),$$

(4.E23) $$R''' = 24A_0x.$$

Now

(4.E24) $$S = 8A_0x^3 + (288K-8A_0)x - 5K[(1)(2)(3)(x-1)^3 + 3(2)(3)(3)(x+1)(x-1)^2 + 3(3)(2)(3)(x+1)^2(x-1)$$

$$+ (1)(2)(3)(x+1)^3]$$

$$= (8A_o - 800K)x^3 + (648K - 8A_o)x.$$

The coefficient T is of the form

(4.E25) $\quad T = B_o x^2 + B_1 x + B_2,$

where

(4.E26) $\quad T(-1) = 12A_o + 144K - 4A_o - 72K(3)(-1-1)^2$

$$= 8A_o - 720K,$$

(4.E27) $\quad T(1) = 8A_o - 720K.$

From equations (4.E25), (4.E26), and (4.E27),

(4.E28) $\quad T(-1) = B_o - B_1 + B_2 = 8A_o - 720K,$

(4.E29) $\quad T(1) = B_o + B_1 + B_2 = 8A_o - 720K.$

Solution of equations (4.E28) and (4.E29) yields

(4.E30) $\quad B_1 = 0,$

(4.E31) $\quad B_2 = 8A_o - B_o - 720K.$

Substitution of equations (4.E30) and (4.E31) into equation (4.E25) yields

(4.E32) $\quad T = B_o x^2 + 8A_o - B_o - 720K.$

Differentiation of equation (4.E32) yields

(4.E33) $\quad T' = 2B_o x.$

Now

$$(4.E34) \qquad U = 2B_0 x - 24A_0 x + 30K(1)(2)(3)(2)(3)(x-1)$$

$$+ 30K(2)(3)(1)(2)(3)(x+1)$$

$$= (2B_0 - 24A_0 + 2160K)x.$$

$$\text{Choose } \frac{A_0}{K} = 72, \quad \frac{B_0}{K} = -184, \quad \frac{\lambda_n}{K} = l_n.$$

Then the differential equation is

$$(4.E35) \qquad (x^6-3x^4+3x^2-1)y_n^{vi} + 18(x^5-2x^3+x)y_n^{v} + (72x^4-72x^2)y_n^{iv}$$

$$+ (-24x^3+72x)y_n^{'''} + (-184x^2+40)y_n^{''} + 64xy_n^{'} + l_n y_n = 0.$$

Solutions of the form

$$(4.E36) \qquad y_n = \sum_{j=0}^{n} a_{jn}x^{n-j}, \quad a_{0n} \neq 0,$$

will be obtained.

$$\begin{array}{r|l}
l_n & y_n = a_{0n}x^n + \cdots + a_{nn}, \quad a \neq 0 \\
64x & y_n^{'} = na_{0n}x^{n-1} + \cdots + a_{n-1,n} \\
-184x^2+40 & y_n^{''} = n(n-1)a_{0n}x^{n-2} + \cdots + 2a_{n-2,n} \\
-24x^3+72x & y_n^{'''} = n(n-1)(n-2)a_{0n}x^{n-3} + \cdots + 6a_{n-3,n} \\
72x^4-72 & y_n^{iv} = n(n-1)(n-2)(n-3)a_{0n}x^{n-4} + \cdots + 24a_{n-4,n} \\
18x^5-36x^3+18x & y_n^{v} = n(n-1)(n-2)(n-3)(n-4)a_{0n}x^{n-5} + \cdots + 120a_{n-5,n} \\
x^6-3x^4+3x^2-1 & y_n^{vi} = n(n-1)(n-2)(n-3)(n-4)(n-5)a_{0n}x^{n-6} + \cdots + 720a_{n-6,n} \\
\hline
\end{array}$$

The highest power of x which appears is n. Since $a_{on} \neq 0$, equating of the coefficient of x^n in the differential equation to zero yields

$$(4.237) \qquad \lambda_n = -64n + 184n(n-1) + 24n(n-1)(n-2) - 72n(n-1)(n-2)(n-3)$$

$$- 18n(n-1)(n-2)(n-3)(n-4) - n(n-1)(n-2)(n-3)(n-4)(n-5).$$

For each choice of n, $n = 0,1,2, \cdots$, the value of λ_n in the differential equation (4.235) is determined from equation (4.237), and a polynomial solution of degree n of the differential equation can be obtained. The determinations of a few of these solutions follow.

Let $n = 0$. Then by equation (4.236),

$$(4.238) \qquad y_o = a_{oo}.$$

Now

$$y_o' = y_o'' = y_o''' = y_o^{iv} = y_o^{v} = y_o^{vi} = \lambda_o = 0,$$

so that

$$0 \cdot a_{oo} = 0.$$

Thus a_{oo} is arbitrary.

Let $n = 1$. Then

$$y_1 = a_{o1}x + a_{11},$$

$$y_1' = a_{o1},$$

$$y_1'' = y_1''' = y_1^{iv} = y_1^{v} = y_1^{vi} = 0,$$

$$\lambda_1 = -64,$$

so that

$$64a_{01}x - 64(a_{01}x + a_{11}) = 0,$$

$$0 \cdot 64a_{01}x - 64a_{11} = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = 0$. Hence

$$(4.539) \qquad y_1 = a_{01}x.$$

Let $n = 2$. Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{iv} = y_2^{v} = y_2^{vi} = 0,$$

$$\ell_2 = -64(2) + 184(2)(1) = 240.$$

Equation of coefficients of like powers of x in the differential equation yields

x^2. $\qquad -368a_{02} + 128a_{02} + 240a_{02} = 0 \cdot a_{02} = 0$

x : $\qquad 64a_{12} + 240a_{12} = 304a_{12} = 0.$

x^0: $\qquad 80a_{02} + 240a_{22} = 0.$

Thus a_{02} is arbitrary, $a_{12} = 0$, and $a_{22} = -\frac{1}{3}a_{02}$. Hence

$$(4.540) \qquad y_2 = a_{02}(x^2 - \tfrac{1}{3}) \ .$$

Let $n = 3$. Then

$$y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{03}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{03}x + 2a_{13},$$

$$y_3''' = 6a_{03},$$

$$y_3^{iv} = y_3^{v} = y_3^{vi} = 0,$$

$$I_3 = -192 + 184(3)(2) + 24(3)(2)(1) = 1056.$$

Equation of coefficients of like powers of x in the differential equation yields

x^3: $$-144a_{03} - 1104a_{03} + 192a_{03} + 1056a_{03} = 0 \cdot a_{03} = 0.$$

x^2: $$-368a_{13} + 128a_{13} + 1056a_{13} = 816a_{13} = 0.$$

x : $$432a_{03} + 240a_{03} + 64a_{23} + 1056a_{23} = 672a_{03} + 1120a_{23} = 0.$$

x^0: $$80a_{13} + 1056a_{33} = 1056a_{33} = 0.$$

Thus a_{03} is arbitrary, $a_{13} = 0$, $a_{23} = -\frac{3}{5}a_{03}$, $a_{33} = 0$. Hence

(4.241) $$y_3 = a_{03}(x^3 - \frac{3}{5}x).$$

Let $n = 4$. Then

$$y_4 = a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44},$$

$$y_4' = 4a_{04}x^3 + 3a_{14}x^2 + 2a_{24} + a_{34},$$

$$y_4'' = 12a_{04}x^2 + 6a_{14}x + 2a_{24},$$

$$y_4''' = 24a_{04}x + 6a_{14},$$

$$y_4^{iv} = 24a_{04},$$

$$l_4 = -256 + 184(4)(3) + 24(4)(3)(2) - 72(4)(3)(2)(1) = 800.$$

Equation of coefficients of like powers of x in the differential equation yields

x^4: $$1728a_{04} - 576a_{04} - 2208a_{04} + 256a_{04} + 800a_{04} = 0 \cdot a_{04} = 0.$$

x^3: $$-144a_{14} - 1104a_{14} = -1248a_{14} = 0.$$

x^2: $$-1728a_{04} + 1728a_{04} - 368a_{24} + 480a_{04} + 128a_{24}$$
$$+ 800a_{24} = 560a_{24} + 480a_{04} = 0.$$

x: $$432a_{14} + 240a_{14} + 64a_{34} + 800a_{34}$$
$$= 1400a_{34} + 672a_{14} = 1400a_{34} = 0.$$

x^0: $$80a_{24} + 800a_{44} = 0.$$

Thus a_{04} is arbitrary, $a_{14} = 0$, $a_{24} = -\frac{6}{7}a_{04}$, $a_{34} = 0$, $a_{44} = -\frac{1}{10}a_{24} = -\frac{1}{10}\left(-\frac{6}{7}a_{04}\right) = \frac{3}{35}a_{04}$. Hence

$$y_4 = a_{04}\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right) \tag{4.242}$$

Further solutions y_n, $n = 5, 6, 7, \cdots$, may be obtained in a similar manner.

These polynomials form an orthogonal system over the fundamental interval $[-1,1]$ with respect to a weight function of unity, and they might be considered as analogous to a set of Jacobi polynomials.

CHAPTER V

SOLUTIONS IN THE SEMI-INFINITE INTERVAL

The Semi-infinite Interval

In this chapter a fundamental interval $[\alpha, \beta]$ which extends to infinity in one direction will be considered. The two possible semi-infinite intervals, $(-\infty, a]$ and $[a, \infty)$, are fundamentally the same, so the discussion will be based on the interval $[a, \infty)$, where a is finite.

The Weight Function $w(x)$

As in Chapter IV, the possibilities of the vanishing of $w(x)$, $w'(x)$, $w''(x)$, $w'''(x)$, and $w^{iv}(x)$ at the finite value $x = a$ must be considered if $w(x)$ is to play its part in the satisfaction of the nine orthogonality conditions. Since in the present discussion the fundamental interval extends to infinity in one direction, it is necessary to consider the limit of a given function of the variable x as it approaches infinity. For the sake of convenience the condition that

$$\lim_{x \to \infty} f(x) = 0$$

will henceforth be denoted by the statement "$f(x) = 0$ at ∞."

The fact that P is a polynomial prohibits the vanishing of P as x approaches ∞. Consequently, orthogonality condition (1)

requires that $w(x)$ vanish as x approaches ∞. Similarly, orthogonality conditions (ii), (iii), (v), and (vi) require, respectively, that $w'(x)$, $w''(x)$, $w'''(x)$, and $w^{iv}(x)$ each vanish as x approaches ∞. Conditions (vii), (viii), and (ix) require that w be a factor in each case of the right member of the identity. Consequently, if w', w'', w''', w^{iv}, and w^{v} are each expressible as the product of w and a rational function in which the denominator divides the numerator, the form of w will be such that the nine orthogonality conditions are satisfied.

Consider a choice of

$$(5.1) \qquad w = e^{-hx}(x-a)^{g}, \quad h > 0,$$

where g and h are real constants. Successive differentiations of equation (5.1) yield:

$$(5.2) \quad w' = -he^{-hx}(x-a)^{g} + ge^{-hx}(x-a)^{g-1}$$

$$= e^{-hx}(x-a)^{g}[-h(x-a) + g]/(x-a),$$

$$(5.3) \quad w'' = h^{2}e^{-hx}(x-a)^{g} - 2ghe^{-hx}(x-a)^{g-1} + (g-1)ge^{-hx}(x-a)^{g-2}$$

$$= e^{-hx}(x-a)^{g}[h^{2}(x-a)^{2} - 2gh(x-a) + (g-1)g]/(x-a)^{2},$$

$$(5.4) \quad w''' = -h^{3}e^{-hx}(x-a)^{g} + 3gh^{2}e^{-hx}(x-a)^{g-1} - 3(g-1)ghe^{-hx}(x-a)^{g-2}$$

$$+ (g-2)(g-1)ge^{-hx}(x-a)^{g-3}$$

$$= e^{-hx}(x-a)^{g}[-h^{3}(x-a)^{3} + 3gh^{2}(x-a)^{2} - 3(g-1)gh(x-a)$$

$$+ (g-2)(g-1)g]/(x-a)^{3},$$

$$(5.5)\quad w^{iv} = h^4e^{-hx}(x-a)^g - 4gh^3e^{-hx}(x-a)^{g-1} + 6(g-1)gh^2e^{-hx}(x-a)^{g-2}$$
$$-4(g-2)(g-1)ghe^{-hx}(x-a)^{g-3} + (g-3)(g-2)(g-1)ge^{-hx}(x-a)^{g-4}$$
$$= e^{-hx}(x-a)^g[h^4(x-a)^4 - 4gh^3(x-a)^3 + 6(g-1)gh^2(x-a)^2$$
$$-4(g-2)(g-1)gh(x-a) + (g-3)(g-2)(g-1)g]/(x-a)^4,$$

$$(5.6)\quad w^{v} = -h^5e^{-hx}(x-a)^g + 5gh^4e^{-hx}(x-a)^{g-1} - 10(g-1)gh^3e^{-hx}(x-a)^{g-2}$$
$$+ 10(g-2)(g-1)gh^2e^{-hx}(x-a)^{g-3} - 5(g-3)(g-2)(g-1)ghe^{-hx}(x-a)^{g-4}$$
$$+ (g-4)(g-3)(g-2)(g-1)ge^{-hx}(x-a)^{g-5}$$
$$= e^{-hx}(x-a)^g[-h^5(x-a)^5 + 5gh^4(x-a)^4 - 10(g-1)gh^3(x-a)^3$$
$$+ 10(g-2)(g-1)gh^2(x-a)^2 - 5(g-3)(g-2)(g-1)gh(x-a)$$
$$+ (g-4)(g-3)(g-2)(g-1)g]/(x-a)^5.$$

Thus, for $h > 0$ and with suitable restrictions on g, $w = e^{-hx}(x-a)^g$ will satisfy the orthogonality conditions.

The Leading Coefficient $P(x)$

Neither the function e^{-hx} nor any of its derivatives vanish at any finite value $x = a$. Thus the contradictions obtained in Chapter IV upon the assumption that $P(x)$ could have $x = a$ as a root of multiplicity less than three will likewise be obtained if the factor $(x-\beta)^h$ in the finite interval expression for the weight function $w(x)$ is replaced by e^{-hx}, since neither $(x-\beta)^h$ nor any of its derivatives vanish at the finite value $x = a$. Consequently,

$F(x) = (x-a)^3 V(x)$, where $V(x)$ is a polynomial of degree three, at most, such that $V(a) \neq 0$.

<u>Application of orthogonality conditions</u>. With the choice of $w = e^{-hx}(x-a)^g$, $h > 0$, and $P = (x-a)^3 V$, condition (ix) requires

$$e^{-hx}(x-a)^g Q = 3[e^{-hx}(x-a)^{g+3}V]'$$

$$= 3[-he^{-hx}(x-a)^{g+3}V + (g+3)e^{-hx}(x-a)^{g+2}V$$

$$+ e^{-hx}(x-a)^{g+3}V'],$$

or

(5.7) $\quad Q = 3(x-a)^2[-h(x-a)V + (g+3)V + (x-a)V'].$

Let $\quad V = C_0x^3 + C_1x^2 + C_2x + C_3.$ Then

(5.8) $\quad Q = 3(x-a)^2[-h(x-a)(C_0x^3 + C_1x^2 + C_2x + C_3)$

$$+ (g+3)(C_0x^3 + C_1x^2 + C_2x + C_3) + (x-a)(3C_0x^2 + 2C_1x + C_2)].$$

The coefficient Q is a polynomial of degree five, at most, so the coefficient of x^6 in the right member of equation (5.8) must vanish. Hence, $-3hC_0 = 0$, which implies $C_0 = 0$ since $h \neq 0$. Then

(5.9) $\quad V = C_1x^2 + C_2x + C_3.$

Orthogonality condition (vii) requires

$$e^{-hx}(x-a)^g S = 2[e^{-hx}(x-a)^g R]' - 5[e^{-hx}(x-a)^{g+3}V]'''$$

$$= -2he^{-hx}(x-a)^g R + 2ge^{-hx}(x-a)^{g-1}R + 2e^{-hx}(x-a)^g R'$$

$$+ 5h^3e^{-hx}(x-a)^{g+3}V - 15(g+3)h^2e^{-hx}(x-a)^{g+2}V$$

$$- 15h^2e^{-hx}(x-a)^{g+3}V' + 15(g+2)(g+3)he^{-hx}(x-a)^{g+1}V$$

$$- 5(g+1)(g+2)(g+3)e^{-hx}(x-a)^{g}V - 15(g+2)(g+3)e^{-hx}(x-a)^{g+1}V'$$

$$+ 30(g+3)he^{-hx}(x-a)^{g+2}V',$$

or

$$(5.10) \quad S = \frac{2gR}{x-a} + 2R' - 2hR - 5(g+1)(g+2)(g+3)V$$

$$+ 15(g+2)(g+3)h(x-a)V - 15(g+2)(g+3)(x-a)V'$$

$$+ 30(g+3)h(x-a)^2V' - 15(g+3)h^2(x-a)^2V$$

$$+ 5h^3(x-a)^3V - 15h^2(x-a)^3V'.$$

The polynomial nature of S requires that $(x-a)$ divide $2gR$ in the right member of equation (5.10), which implies

$$(5.11) \qquad R(a) = 0, \quad \text{for } g \neq 0.$$

From equation (5.8), $V = C_1x^2 + C_2x + C_3$, and hence $V' = 2C_1x + C_2$. Let $R = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$, so that $R' = 4A_0x^3 + 3A_1x^2 + 2A_2x + A_3$. Then equation (5.10) becomes

$$(5.12) \quad S = 2g(A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4)/(x-a) + 8A_0x^3 + 6A_1x^2$$

$$+ 4A_2x + 2A_3 - 2hA_0x^4 - 2hA_1x^3 - 2hA_2x^2 - 2hA_3x - 2hA_4$$

$$- 5(g+1)(g+2)(g+3)(C_1x^2 + C_2x + C_3)$$

$$+ 15(g+2)(g+3)h(x-a)(C_1x^2 + C_2x + C_3)$$

$$- 15(g+2)(g+3)(x-a)(2C_1x + C_2) + 30(g+3)h(x-a)^2(2C_1x + C_2)$$

$$- 15(g+3)h^2(x-a)^2(C_1x^2 + C_2x + C_3)$$

$$+ 5h^3(x-a)^3(C_1x^2 + C_2x + C_3) - 15h^2(x-a)^3(2C_1x + C_2).$$

Since S is a polynomial of degree three, at most, the coefficients of x^5 and x^4 in the right member of equation (5.12) must vanish. If the coefficient of x^5 vanishes, then $5h^3C_1 = 0$, which implies $C_1 = 0$ since $h > 0$. Thus from equation (5.9),

(5.13) $V = C_2x + C_3.$

Now if the coefficient of x^4 vanishes, then $-2hA_0 + 5h^3C_2 = 0$ Since $h > 0$, this implies $A_0 = \frac{5}{2}h^2C_2$, so that

(5.14) $R = \frac{5}{2}h^2C_2x^4 + A_1x^3 + A_2x^2 + A_3x + A_4.$

Orthogonality condition (viii) requires

$$e^{-hx}(x-a)^gU = [e^{-hx}(x-a)^gT]' - [e^{-hx}(x-a)^gR]''' + 3[e^{-hx}(x-a)^{g+3}V]^{v}$$

$$= -he^{-hx}(x-a)^gT + ge^{-hx}(x-a)^{g-1}T + e^{-hx}(x-a)^gT'$$

$$+ h^3e^{-hx}(x-a)^gR - 3gh^2e^{-hx}(x-a)^{g-1}R - 3h^2e^{-hx}(x-a)^gR'$$

$$+ 3(g-1)ghe^{-hx}(x-a)^{g-2}R - (g-2)(g-1)ge^{-hx}(x-a)^{g-3}R$$

$$- 3(g-1)ge^{-hx}(x-a)^{g-2}R' + 6ghe^{-hx}(x-a)^{g-1}R' + 3he^{-hx}(x-a)^gR''$$

$$- 3ge^{-hx}(x-a)^{g-1}R'' - e^{-hx}(x-a)^gR''' - 3h^5e^{-hx}(x-a)^{g+3}V$$

$$+ 15(g+3)h^4e^{-hx}(x-a)^{g+2}V + 15h^4e^{-hx}(x-a)^{g+3}V'$$

$$- 30(g+2)(g+3)h^3e^{-hx}(x-c)^{g+1}V - 60(g+3)h^3e^{-hx}(x-c)^{g+2}V'$$

$$+ 30(g+1)(g+2)(g+3)h^2e^{-hx}(x-c)^{g}V + 90(g+2)(g+3)h^2e^{-hx}(x-c)^{g+1}V'$$

$$- 15g(g+1)(g+2)(g+3)he^{-hx}(x-c)^{g-1}V$$

$$- 60(g+1)(g+2)(g+3)he^{-hx}(x-c)^{g}V'$$

$$+ 3(g-1)g(g+1)(g+2)(g+3)e^{-hx}(x-c)^{g-2}V$$

$$+ 15g(g+1)(g+2)(g+3)e^{-hx}(x-c)^{g-1}V',$$

or

$$U = T' - hT + h^3R - 3h^2R' + 3hR'' - R''' - 3h^5(x-c)^3V \tag{5.15}$$

$$+ 15(g+3)h^4(x-c)^2V + 15h^4(x-c)^3V' - 30(g+2)(g+3)h^3(x-c)V$$

$$- 60(g+3)h^3(x-c)^2V' + 30(g+1)(g+2)(g+3)h^2V + 90(g+2)(g+3)h^2(x-c)V'$$

$$- 60(g+1)(g+2)(g+3)hV' + [gT - 3gh^2R + 6ghR' - 3gR''$$

$$- 15g(g+1)(g+2)(g+3)hV + 15g(g+1)(g+2)(g+3)V']/(x-c)$$

$$+ [3(g-1)ghR - 3(g-1)gR' + 3(g-1)g(g+1)(g+2)(g+3)V]/(x-c)^2$$

$$- (g-2)(g-1)gR/(x-c)^3.$$

From equation (5.13), $V = C_2x + C_3$, and hence $V' = C_2$. From equation (5.14), $R = \frac{5}{2}h^2C_2x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$, and hence $R' = 10h^2C_2x^3 + 3A_1x^2 + 2A_2x + A_3$, $R'' = 30h^2C_2x^2 + 6A_1x + 2A_2$, $R''' = 60h^2C_2x + 6A_1$. Let $T = B_0x^2 + B_1x + B_2$, so that $T' = 2B_0x + B_1$. Then equation (5.15) becomes

$$
\begin{aligned}
(5.16)\quad U = {} & 2B_0x + B_1 - h(B_0x^2 + B_1x + B_2) + h^3(\tfrac{5}{2}h^2C_2x^4 + A_1x^3 \\
& + A_2x^2 + A_3x + A_4) - 3h^2(10h^2C_2x^3 + 3A_1x^2 + 2A_2x + A_3) \\
& + 3h(30h^2C_2x^2 + 6A_1x + 2A_2) - (60h^2C_2x + 6A_1) \\
& - 3h^5(x-\alpha)^3(C_2x + C_3) + 15(g+3)h^4(x-\alpha)^2(C_2x + C_3) + 15C_2h^4(x-\alpha)^3 \\
& - 30(g+2)(g+3)h^3(x-\alpha)(C_2x + C_3) - 60C_2(g+3)h^3(x-\alpha)^2 \\
& + 30(g+1)(g+2)(g+3)h^2(C_2x + C_3) + 30C_2(g+2)(g+3)h^2(x-\alpha) \\
& - 60C_2(g+1)(g+2)(g+3)h + [g(B_0x^2 + B_1x + B_2) - 3gh^2(\tfrac{5}{2}h^2C_2x^4 \\
& + A_1x^3 + A_2x^2 + A_3x + A_4) + 6gh(10h^2C_2x^3 + 3A_1x^2 + 2A_2x + A_3) \\
& - 3g(30h^2C_2x^2 + 6A_1x + 2A_2) - 15g(g+1)(g+2)(g+3)h(C_2x + C_3) \\
& + 15C_2g(g+1)(g+2)(g+3)]/(x-\alpha) + [3(g-1)gh(\tfrac{5}{2}h^2C_2x^4 \\
& + A_1x^3 + A_2x^2 + A_3x + A_4) - 3(g-1)g(10h^2C_2x^3 + 3A_1x^2 + 2A_2x + A_3) \\
& + 3(g-1)g(g+1)(g+2)(g+3)(C_2x + C_3)]/(x-\alpha)^2 - (g-2)(g-1)g(\tfrac{5}{2}h^2C_2x^4 \\
& + A_1x^3 + A_2x^2 + A_3x + A_4)/(x-\alpha)^3.
\end{aligned}
$$

Since U is a first degree polynomial, the coefficients of x^4, x^3, and x^2 in the right member of equation (5.16) must vanish. If the coefficient of x^4 vanishes, then $\frac{5}{2}h^5C_2 - 3h^5C_2 = 0$, which implies $C_2 = 0$ since $h > 0$. Thus from equation (5.13),

$$(5.17)\qquad V = C_3 \neq 0;$$

and from equation (5.14),

$$(5.18) \qquad R = A_1x^3 + A_2x^2 + A_3x + A_4.$$

Now if the coefficient of x^5 vanishes, then $h^3A_1 - 3h^5C_5 = 0$, which implies $A_1 = 3h^2C_5$ since $h > 0$. Then equation (5.18) becomes

$$(5.19) \qquad R = 3C_5h^2x^3 + A_2x^2 + A_3x + A_4.$$

Now if the coefficient of x^2 vanishes, then $-hS_0 + h^3A_2 - 27h^4C_5 + 3ah^5C_5 + 15(g+3)h^4C_5 - 8gh^4C_5 = 0$; or since $h > 0$, $S_0 = h^2A_2 + 3(2g + 3ha + 6)h^3C_5$. Hence,

$$(5.20) \quad T = [h^2A_2 + 3(2g+3ha+6)h^3C_5]x^2 + S_1x + S_2.$$

Equation (5.15) may now be written

$$\begin{aligned}(5.21) \quad U = {} & T' - hT + h^3R - 3h^2R' + 3hR'' - 18h^2C_5 - 3C_5h^5(x-a)^3 \\ & + 15C_5(g+3)h^4(x-a)^2 - 30C_5(g+2)(g+3)h^3(x-a) \\ & + 30C_5(g+1)(g+2)(g+3)h^2 + [gT - 3gh^2R + 6ghR' - 3gR'' \\ & - 15C_5g(g+1)(g+2)(g+3)h]/(x-a) + [3(g-1)ghR - 3(g-1)gR' \\ & + 3C_5(g-1)g(g+1)(g+2)(g+3)]/(x-a)^2 - (g-2)(g-1)gR/(x-a)^3.\end{aligned}$$

For convenience, let the numerators of the fractional terms in the right member of equation (5.21) be designated as follows:

$$(5.22) \quad Y_1(x) = gT - 3gh^2R + 6ghR' - 3gR'' - 15C_5g(g+1)(g+2)(g+3)h,$$

$$(5.23) \quad Y_2(x) = 3(g-1)ghR - 3(g-1)gR' + 3C_5(g-1)g(g+1)(g+2)(g+3),$$

$$(5.24) \qquad N_g = (g-2)(g-1)g.$$

The fractional terms in the right member of equation (5.21) may be combined to yield the single term

$$(5.25) \quad [(x-e)^2 Y_1 + (x-e)Y_2 - N_g R]/(x-e)^3.$$

The polynomial nature of U requires that the numerator of the term represented by expression (5.25) be divisible by the denominator $(x-e)^3$ of the term. A necessary and sufficient condition that this be true is that the numerator and the first and second derivatives of the numerator vanish at $x = e$. If the numerator of expression (5.25) vanishes at $x = e$, then $N_g R(e) = (g-2)(g-1)gR(e) = 0$. But equation (5.11) has already imposed the condition $R(e) = 0$ for $g \neq 0$, so no new restriction is obtained. If the first derivative of the numerator of (5.25) vanishes at $x = e$, then $Y_2(e) - N_g R'(e) = 0$, or

$$(5.26) \qquad N_g R'(e) = Y_2(e).$$

If the second derivative of the numerator of (5.25) vanishes at $x = e$, then $2Y_1(e) + 2Y_2'(e) - N_g R''(e) = 0$, or

$$(5.27) \qquad N_g R''(e) = 2[Y_1(e) + Y_2'(e)].$$

Substitution of the expressions for Y_1, Y_2, and N_g from equations (5.22), (5.23), and (5.24) into equations (5.26) and (5.27) yields, after dividing by $(g-1)g$ and g, respectively, and making use of equation (5.11):

$$(5.28) \quad (g-3)R'(e) = -3C_3(g+1)(g+2)(g+3) \qquad \text{for } g \neq 0, \ g \neq 1;$$

(5.29) $T(a) = \frac{1}{2}(g+1)(g+2)R''(a) - 3(g+1)hR'(a)$

$+ 15C_3(g+1)(g+2)(g+3)h \quad \text{for} \quad g \neq 0.$

Now $g = 3$ implies a contradiction in equation (5.28), since from equation (5.17) $C_3 \neq 0$. Hence

(5.30) $g \neq 3$.

In view of equation (5.17), equations (5.7) and (5.10) may be rewritten, respectively,

(5.31) $Q = 3C_3(x-a)^2[(g+3) - h(x-a)]$,

(5.32) $S = 2gR/(x-a) + 2R' - 2hR - 3C_3(g+1)(g+2)(g+3)$

$+ 15C_3(g+2)(g+3)h(x-a) - 15C_3(g+3)h^2(x-a)^2$

$+ 5C_3h^3(x-a)^3.$

Orthogonality condition (i) requires

(5.33) $wP = C_3e^{-hx}(x-a)^{g+3} = 0 \quad \text{at} \quad x = a \quad \text{and at} \quad \infty$.

Since $h > 0$, the condition is satisfied at ∞. If wP is to vanish at $x = a$, then with $C_3 \neq 0$ it is necessary that

(5.34) $g > -3$.

Condition (ii) requires

(5.35) $(wP)' = C_3[-he^{-hx}(x-a)^{g+3} + (g+3)e^{-hx}(x-a)^{g+2}]$

$$= C_3 e^{-hx}(x-a)^{g+2}[-h(x-a) + (g+3)] = 0 \quad \text{at} \quad x = a \text{ and at } \infty.$$

Since $h > 0$, the condition is satisfied at ∞. If $g > -2$, the condition is satisfied at $x = a$. However if $-3 < g \leq -2$, then $(g+3) = 0$, which is impossible. Hence

$$(5.36) \qquad g > -2.$$

Condition (iii) requires

$$(5.37) \quad (wF)'' = C_3[h^2 e^{-hx}(x-a)^{g+3} - 2(g+3)he^{-hx}(x-a)^{g+2}$$

$$+ (g+2)(g+3)e^{-hx}(x-a)^{g+1}]$$

$$= C_3 e^{-hx}(x-a)^{g+1}[h^2(x-a)^2 - 2(g+3)h(x-a) + (g+2)(g+3)]$$

$$= 0 \quad \text{at} \quad x = a \quad \text{and at} \quad \infty$$

Since $h > 0$, the condition is satisfied at ∞. If $g > -1$, the condition is satisfied at $x = a$. However if $-2 < g \leq -1$, then $(g+2)(g+3) = 0$, which is impossible. Hence

$$(5.38) \qquad g > -1.$$

Condition (iv) requires

$$(5.39) \quad wR - (wF)'' = e^{-hx}(x-a)^g R - (wF)''$$

$$= e^{-hx}(x-a)^g \{R - C_3(x-a)[h^2(x-a)^2$$

$$- 2(g+3)h(x-a) + (g+2)(g+3)]\} = 0 \quad \text{at} \quad x = a \quad \text{and at} \quad \infty$$

Since $h > 0$, the condition is satisfied at ∞. If $g > 0$, the condition is satisfied at $x = a$. However if $-1 < g \leq 0$, then the condition is satisfied at $x = a$ only if $R(a) = 0$. But from equation (5.11), $R(a) = 0$ for $g \neq 0$. Hence

$$(5.40) \qquad R(a) = 0.$$

Condition (v) requires

$$\begin{aligned}(5.41) \quad (wR)' - 3(wP)''' = {} & -he^{-hx}(x-a)^g R + ge^{-hx}(x-a)^{g-1}R \\ & + e^{-hx}(x-a)^g R' + C_g[3h^3e^{-hx}(x-a)^{g+3} \\ & - 9(g+3)h^2e^{-hx}(x-a)^{g+2} \\ & + 9(g+2)(g+3)he^{-hx}(x-a)^{g+1} - 3(g+1)(g+2)(g+3)e^{-hx}(x-a)^g] \\ = {} & e^{-hx}(x-a)^{g-1}\Big[gR + (x-a)\{R' - hR \\ & - 3C_g(g+1)(g+2)(g+3) + C_g(x-a)[9(g+2)(g+3)h \\ & - 9(g+3)h^2(x-a) + 3h^3(x-a)^2]\}\Big] = 0 \text{ at } x = a \text{ and at } \infty.\end{aligned}$$

Since $h > 0$, the condition is satisfied at ∞. For $g > 1$, the condition is satisfied at $x = a$. If $0 < g \leq 1$, then still the condition is satisfied at $x = a$, since by equation (5.40) $R(a) = 0$. If $-1 < g \leq 0$, then the condition is satisfied only if, in addition, $R'(a) - hR(a) - 3C_g(g+1)(g+2)(g+3) = 0$, or

$$(5.42) \qquad R'(a) = 3C_g(g+1)(g+2)(g+3), \quad \text{for } -1 < g \leq 0.$$

Now equations (5.28) and (5.42) together imply $-1/(g-5) = 1$ for $-1 < g < 0$, which is impossible. Thus the orthogonality conditions cannot be satisfied for $-1 < g < 0$. In view of equation (5.38), then,

$$(5.43) \qquad g \geq 0.$$

For $g = 0$, equation (5.42) becomes

$$(5.44) \qquad K'(a) = 18C_5, \quad \text{for } g = 0.$$

Condition (vi) requires

$$(5.45) \quad wT - (wR)'' + 3(wP)^{iv} = e^{-hx}(x-a)^{g}T - h^2e^{-hx}(x-a)^{g}R$$

$$+ 2ghe^{-hx}(x-a)^{g-1}R - (g-1)ge^{-hx}(x-a)^{g-2}R$$

$$+ 2he^{-hx}(x-a)^{g}R' - 2ge^{-hx}(x-a)^{g-1}R' - e^{-hx}(x-a)^{g}R''$$

$$+ 3C_5h^4e^{-hx}(x-a)^{g+3} - 12C_5(g+3)h^3e^{-hx}(x-a)^{g+2}$$

$$+ 18C_5(g+2)(g+3)h^2e^{-hx}(x-a)^{g+1}$$

$$- 12C_5(g+1)(g+2)(g+3)he^{-hx}(x-a)^{g}$$

$$+ 3C_5g(g+1)(g+2)(g+3)e^{-hx}(x-a)^{g-1}$$

$$= e^{-hx}(x-a)^{g-2}\Big[-(g-1)gR$$

$$+ (x-a)\Big[-2gR' + 2ghR + 3C_5g(g+1)(g+2)(g+3)$$

$$+ (x-a)\{T - h^2R + 2hR' - R'' - 12C_5(g+1)(g+2)(g+3)h$$

$$+ (x-a)[18C_5(g+2)(g+3)h^2 - 12C_5(g+3)h^3(x-a)$$

$$+ 3C_3h^4(x-a)^2]\}\Big]\Big| = 0 \quad \text{at} \quad x = a \quad \text{and at } \infty.$$

Since $h > 0$, the condition is satisfied at ∞. For $g > 2$, the condition is satisfied at $x = a$. If $1 < g \leq 2$, then still the condition is satisfied at $x = a$, since by equation (5.40) $R(a) = 0$. If $0 < g \leq 1$, the condition is satisfied only if, in addition, $-2gR'(a) + 2ghR(a) + 3C_3g(g+1)(g+2)(g+3) = 0$, or

$$(5.46) \qquad R'(a) = \frac{3}{2}C_3(g+1)(g+2)(g+3), \quad \text{for } 0 < g \leq 1.$$

Now equations (5.28) and (5.46) together imply $-1/(g-5) = 1/2$ for $0 < g < 1$, which is impossible. Thus the orthogonality conditions cannot be satisfied for $0 < g < 1$. In view of equation (5.45), then,

$$(5.47) \qquad g = 0 \quad \text{or} \quad g \geq 1.$$

For $g = 1$, equation (5.46) becomes

$$(5.48) \qquad R'(a) = 36C_3.$$

If $g = 0$, condition (vi) is satisfied only if $T(a) - h^2R(a) + 2hR'(a) - R''(a) - 12C_3(g+1)(g+2)(g+3)h = 0$, or

$$(5.49) \qquad T(a) = 72C_3h - 2hR'(a) + R''(a), \quad \text{for } g = 0.$$

Combination of equations (5.30) and (5.47) yields

$$(5.50) \qquad g = 0 \quad \text{or} \quad 1 \leq g < 5 \quad \text{or} \quad g > 5.$$

Equations (5.19), (5.20), (5.21), (5.28), (5.29), (5.31), (5.32), (5.40), (5.44), (5.48), (5.49), and (5.50) constitute the totality of restrictions imposed on coefficients Q, R, S, T, and U by the nine orthogonality conditions with $P = C_3(x-a)^3$, $C_3 \neq 0$, and $w = e^{-hx}(x-a)^g$, $h > 0$. In the summary C_3 will be replaced by K.

The Interval $(-\infty, a]$

The foregoing treatment made use of the fact that $h \neq 0$. If $h < 0$ (rather than $h > 0$) in the weight function $w = e^{-hx}(x-a)^g$, the preceding development can be reproduced to yield comparable results for the interval $(-\infty, a]$. With this interval under consideration, the condition that

$$\lim_{x \to -\infty} f(x) = 0$$

would be denoted by the statement "$f(x) = 0$ at $-\infty$."

Summary

The choice of $w(x) = e^{-hx}(x-a)^g$, where $h > 0$ [or $h < 0$] and $g = 0$ or $1 \leq g < 5$ or $g > 5$, will serve as a weight function in the orthogonalization of the solution set $\{y_n\}$, $n = 0, 1, 2, \cdots$, represented by equation (1.2) of the differential equation (1.23) over the fundamental interval $[a, \infty)$ [or $(-\infty, a]$].

This choice of $w(x)$ imposes the following restrictions on the coefficients of equation (1.23) in the satisfaction of the nine orthogonality conditions:

$P = K(x-c)^3$, $\quad K$ a non-zero constant,

$Q = 3K(x-c)^2[(g+3) - h(x-c)]$;

$R = 3Kh^2x^3 + A_2x^2 + A_3x + A_4$, $\quad A_i$ = constant, $i = 2,3,4$,

$$R(c) = 0,$$

$$R'(c) = -\frac{3K}{g-3}(g+1)(g+2)(g+3) \quad \text{for } g \neq 0, \; g \neq 1,$$

$$R'(c) = 15K \quad \text{for } g = 0,$$

$$R'(c) = 36K \quad \text{for } g = 1;$$

$$S = 2gR/(x-c) + 2R' - 2hR - 3K(g+1)(g+2)(g+3)$$

$$+ 15K(g+2)(g+3)h(x-c) - 15K(g+3)h^2(x-c)^2$$

$$+ 5Kh^3(x-c)^3;$$

$$T = [h^2A_2 + 3K(2g+3hc+6)h^3]x^2 + B_1x + B_2,$$

$$B_i = \text{constant}, \quad i = 1,2,$$

$$T(c) = \frac{1}{2}(g+1)(g+2)R''(c) - 3(g+1)hR'(c)$$

$$+ 15K(g+1)(g+2)(g+3)h \quad \text{for } g \neq 0,$$

$$T(c) = 72Kh - 2hR'(c) + R''(c) \quad \text{for } g = 0;$$

$$U = T' - hT + h^3R - 3h^2R' + 3hR'' - 18Kh^2 - 3Kh^5(x-c)^3$$

$$+ 15K(g+3)h^4(x-c)^2 - 30K(g+2)(g+3)h^3(x-c)$$

$$+ 30K(g+1)(g+2)(g+3)h^2 + [gT - 3gh^2R + 6ghR'$$

$$- 3gR'' - 15Kg(g+1)(g+2)(g+3)h]/(x-c)$$

$$+ [3(g-1)ghR - 3(g-1)gR' + 3K(g-1)g(g+1)(g+2)(g+3)]/(x-c)^2$$

$$- (g-2)(g-1)gR/(x-c)^3.$$

Example of the Semi-infinite Interval

Choose $h = 1$, $g = 0$, $e = 0$, and consider the interval $[e, \infty)$. Then

(5.E1) $w = e^{-x}$.

(5.E2) $P = Kx^3$, $K \neq 0$.

(5.E3) $Q = 3Kx^2(3-x) = 3K(3x^2 - x^3)$

(5.E4) $R = 3Kx^3 + A_2x^2 + A_3x + A_4$.

where

(5.E5) $R(0) = A_4 = 0$,

(5.E6) $R'(0) = A_3 = 18K$.

(5.E7) $S = 2(9Kx^2 + 2A_2x + 18K) - 2(3Kx^3 + A_2x^2 + 18Kx)$

$$- 30K + 90Kx - 45Kx^2 + 5Kx^3$$

$$= -Kx^3 - (27K + 2A_2)x^2 + (54K + 4A_2)x + 6K.$$

(5.E8) $T = (A_2 + 18K)x^2 + B_1x + B_2$,

where

(5.E9) $T(0) = B_2 = 72K - 36K + 2A_2 = 36K + 2A_2$.

(5.E10) $U = 2(A_2 + 18K)x + B_1 - (A_2 + 18K)x^2 - B_1x - 36K - 2A_2$

$$+ 3Kx^3 + A_2x^2 + 18Kx - 27Kx^2 - 6A_2x - 54K$$

$$+ 54Kx + 6A_2 - 18K - 3Kx^3 + 45Kx^2 - 180Kx + 180K$$

$$= -(72K + 4A_2 + B_1)x + (72K + 4A_2 + B_1).$$

The differential equation is thus

$$Kx^3y_n^{vi} + 3K(3x^2 - x^3)y_n^{v} + (3Kx^3 + A_2x^2 + 18Kx)y_n^{iv}$$

$$+ [-Kx^3 - (27K + 2A_2)x^2 + (54K + 4A_2)x + 6K]y_n'''$$

$$+ [(A_2 + 18K)x^2 + B_1x + (36K + 2A_2)]y_n'' + [-(72K + 4A_2 + B_1)x$$

$$+ (72K + 4A_2 + B_1)]y_n' + \lambda_n y_n = 0,$$

or, upon division by $K \neq 0$,

$$\text{(5.E11)} \qquad x^3y_n^{vi} + 3(3x^2 - x^3)y_n^{v} + (3x^3 + \frac{A_2}{K}x^2 + 18x)y_n^{iv}$$

$$+ [-x^3 - (27 + 2\frac{A_2}{K})x^2 + (54 + 4\frac{A_2}{K})x + 6]y_n'''$$

$$+ [(\frac{A_2}{K} + 18)x^2 + \frac{B_1}{K}x + (36 + 2\frac{A_2}{K})]y_n''$$

$$+ [-(72 + 4\frac{A_2}{K} + \frac{B_1}{K})x + (72 + 4\frac{A_2}{K} + \frac{B_1}{K})]y_n'$$

$$+ \frac{\lambda_n}{K}y_n = 0$$

Choose $A_2 = -18K$, $B_1 = K$, $\frac{\lambda_n}{K} = l_n$. Then equation (5.E11) becomes

$$\text{(5.E12)} \qquad x^3y_n^{vi} + (-3x^3 + 9x^2)y_n^{v} + (3x^3 - 18x^2 + 18x)y_n^{iv}$$

$$+ (-x^3 + 9x^2 - 18x + 6)y_n''' + xy_n''$$

$$+ (-x + 1)y_n' + l_ny_n = 0.$$

Solutions of the form

$$(5.E13) \qquad y_n = \sum_{j=0}^{n} a_{jn} x^{n-j}, \quad a_{on} \neq 0,$$

will be obtained.

l_n	$y_n = a_{on}x^n + \cdots + a_{nn}, \quad a_{on} \neq 0$
$-x+1$	$y_n' = na_{on}x^{n-1} + \cdots + a_{n-1,n}$
x	$y_n'' = n(n-1)a_{on}x^{n-2} + \cdots + 2a_{n-2,n}$
$-x^3+9x^2-18x+6$	$y_n''' = n(n-1)(n-2)a_{on}x^{n-3} + \cdots + 6a_{n-3,n}$
$3x^3-18x^2+18x$	$y_n^{iv} = n(n-1)(n-2)(n-3)a_{on}x^{n-4} + \cdots + 24a_{n-4,n}$
$-3x^3+9x^2$	$y_n^{v} = n(n-1)(n-2)(n-3)(n-4)a_{on}x^{n-5} + \cdots + 120a_{n-5,n}$
x^3	$y_n^{vi} = n(n-1)(n-2)(n-3)(n-4)(n-5)a_{on}x^{n-6} + \cdots + 720a_{n-6,n}$

The highest power of x which appears is n. The coefficient of x^n is $[l_n - n - n(n-1)(n-2)]a_{on} = 0$. Since $a_{on} \neq 0$, then

$$(5.E14) \qquad l_n = n + n(n-1)(n-2).$$

For each choice of n, $n = 0,1,2, \cdots$, the value of l_n is determined from equation (5.E14), and a polynomial solution of degree n of equation (5.E12) can be obtained. The determinations of a few of these solutions follow

<u>Let $n = 0$.</u> Then

(5.E15) $$y_0 = a_{00},$$

$$y_0' = y_0'' = y_0''' = y_0^{iv} = y_0^{v} = y_0^{vi} = l_0 = 0,$$

so that equation (5.E12) becomes

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary.

<u>Let $n = 1$.</u> Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{iv} = y_1^{v} = y_1^{vi} = 0,$$

$$l_1 = 1,$$

so that equation (5.E12) becomes

$$(-x+1)a_{01} + a_{01}x + a_{11} = 0 \cdot a_{01}x + a_{01} + a_{11} = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = -a_{01}$. Consequently

(5.E16) $$y_1 = a_{01}x - a_{01} = a_{01}(x-1).$$

<u>Let $n = 2$.</u> Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{o2},$$

$$y_2''' = y_2^{iv} = y_2^{v} = y_2^{vi} = 0,$$

$$l_2 = 2,$$

so that equation (5.E12) becomes

$$2a_{o2}x + (-x+1)(2a_{o2}x + a_{12}) + 2(a_{o2}x^2 + a_{12}x + a_{22})$$

$$= 0 \quad a_{o2}x^2 + (4a_{o2} + a_{12})x + (a_{12} + 2a_{22}) = 0.$$

Thus, a_{o2} is arbitrary, $a_{12} = -4a_{o2}$, $a_{22} = -a_{12}/2 = 2a_{o2}$.

Consequently,

(5.E17) $$y_2 = a_{o2}x^2 - 4a_{o2}x + 2a_{o2} = a_{o2}(x^2 - 4x + 2).$$

<u>Let $n = 3$.</u> Then

$$y_3 = a_{o3}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{o3}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{o3}x + 2a_{13},$$

$$y_3''' = 6a_{o3},$$

$$y_3^{iv} = y_3^{v} = y_3^{vi} = 0,$$

$$l_3 = 3 + 3(2)(1) = 9.$$

Equating of like powers of x in equation (5.E12) yields

x^3: $\quad -6a_{03} - 3a_{03} + 9a_{03} = 0 \cdot a_{03} = 0,$

x^2: $\quad 54a_{03} + 6a_{03} - 2a_{13} + 3a_{03} + 9a_{13} = 63a_{03} + 7a_{13} = 0$

x : $\quad -108a_{03} + 2a_{13} - a_{23} + 2a_{13} + 9a_{23} = -108a_{03} + 4a_{13} + 8a_{23} = 0,$

x^0: $\quad 36a_{03} + a_{23} + 9a_{33} = 0.$

Thus a_{03} is arbitrary, $a_{13} = -9a_{03}$, $a_{23} = (108a_{03} - 4a_{13})/8$
$= (108a_{03} + 36a_{03})/8 = 18a_{03}$, $a_{33} = (-36a_{03} - a_{23})/9$
$= (-36a_{03} - 18a_{03})/9 = -6a_{03}$. Consequently,

$$y_3 = a_{03}x^3 - 9a_{03}x^2 + 18a_{03}x - 6a_{03} \tag{5.E13}$$

$$= a_{03}(x^3 - 9x^2 + 18x - 6).$$

Let $n = 4$. Then

$$y_4 = a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44},$$

$$y_4' = 4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34},$$

$$y_4'' = 12a_{04}x^2 + 6a_{14}x + 2a_{24},$$

$$y_4''' = 24a_{04}x + 6a_{14},$$

$$y_4^{iv} = 24a_{04},$$

$$y_4^{v} = y_4^{vi} = 0,$$

$$\lambda_4 = 4 + 4(3)(2) = 28.$$

Equation of like powers of x in equation (5.E12) yields

x^4: $\quad -24a_{04} - 4a_{04} + 28a_{04} = 0 \cdot a_{04} = 0,$

x^3: $\quad 72a_{04} - 6a_{14} + 216a_{04} + 12a_{04} - 3a_{14} + 4a_{04} + 28a_{14}$

$$= 19a_{14} + 304a_{04} = 0,$$

x^2: $\quad -432a_{04} + 54a_{14} - 432a_{04} + 6a_{14} - 2a_{24} + 3a_{14} + 28a_{24}$

$$= -864a_{04} + 63a_{14} + 26a_{24} = 0,$$

x: $\quad 432a_{04} - 108a_{14} + 144a_{04} + 2a_{24} - a_{34} + 2a_{24} + 28a_{34}$

$$= 576a_{04} - 108a_{14} + 4a_{24} + 27a_{34} = 0,$$

x^0: $\quad 36a_{14} + a_{34} + 28a_{44} = 0.$

Thus a_{04} is arbitrary, $a_{14} = -16a_{04}$, $a_{24} = (864a_{04} - 63a_{14})/26$ $= [864a_{04} - 63(-16a_{04})]/26 = 72a_{04}$, $a_{34} = (-576a_{04} + 108a_{14}$ $-4a_{24})/27 = [-576a_{04} + 108(-16a_{04}) - 4(72a_{04})]/27 = -96a_{04}$, $a_{44} = (-36a_{14} - a_{34})/28 = [-36(-16a_{04}) - (-96a_{04})]/28 = 24a_{04}$. Consequently,

$$y_5 = a_{04}x^4 - 16a_{04}x^3 + 72a_{04}x^2 - 96a_{04}x + 24a_{04} \tag{5.E13}$$

$$= a_{04}(x^4 - 16x^3 + 72x^2 - 96x + 24).$$

Further solutions y_n, $n = 5, 6, 7, \cdots$, can be determined in a similar manner

The solution set obtained forms an orthogonal system, with respect to the weight function e^{-x}, over the interval $[0, \infty)$. This set might be considered as analogous to a set of Laguerre polynomials.

CHAPTER VI

SOLUTIONS IN THE INFINITE INTERVAL

The Infinite Interval

The fundamental interval $[\alpha,\beta]$ may extend to infinity in both directions. As in the finite and semi-infinite intervals, the choice of the weight function $w(x)$ will determine the forms of the coefficients P, Q, R, S, T, and U of the differential equation (1.23).

The Weight Function $w(x)$

Since the coefficients P, Q, R, S, T, and U are polynomials, they do not vanish at $\pm\infty$. Thus, orthogonality conditions (i), (ii), (iii), (v), and (vi) require, respectively, that $w(x)$, $w'(x)$, $w''(x)$, $w'''(x)$, and $w^{iv}(x)$ each vanish at $\pm\infty$. Conditions (vii), (viii), and (ix) again require that w be a factor in each case of the right member of the identity, which implies that w', w'', w''', w^{iv}, and w^{v} must again each be expressible as the product of w and a rational function in which the denominator divides the numerator.

Consider a choice of

$$(6.1) \qquad w = e^{-hx^2}, \quad h > 0,$$

where h is a real constant. Successive differentiations of equation (6.1) yield

$$(6.2) \qquad w' = -2hxe^{-hx^2},$$

$$(6.3) \qquad w'' = 4h^2x^2e^{-hx^2} - 2he^{-hx^2}$$

$$= 2h(2hx^2 - 1)e^{-hx^2},$$

$$(6.4) \qquad w''' = -8h^3x^3e^{-hx^2} + 12h^2xe^{-hx^2}$$

$$= 4h^2x(-2hx^2 + 3)e^{-hx^2},$$

$$(6.5) \qquad w^{iv} = 16h^4x^4e^{-hx^2} - 48h^3x^2e^{-hx^2} + 12h^2e^{-hx^2}$$

$$= 4h^2(4h^2x^4 - 12hx^2 + 3)e^{-hx^2},$$

$$(6.6) \qquad w^{v} = -32h^5x^5e^{-hx^2} + 160h^4x^3e^{-hx^2} - 120h^3xe^{-hx^2}$$

$$= 8h^3x(-4h^2x^4 + 20hx^2 - 15)e^{-hx^2}.$$

Thus, for $h > 0$, a weight function $w = e^{-hx^2}$ will satisfy the nine orthogonality conditions.

Application of orthogonality conditions. With the choice of $w = e^{-hx^2}$, condition (vii) requires

$$e^{-hx^2}S = 2(e^{-hx^2}R)' - 5(e^{-hx^2}P)'''$$

$$= 2(-2hxe^{-hx^2}R + e^{-hx^2}R') - 5(12h^2xe^{-hx^2}P$$

$$-6he^{-hx^2}P' - 8h^3x^3e^{-hx^2}P + 12h^2x^2e^{-hx^2}P'$$

$$-6hxe^{-hx^2}P'' + e^{-hx^2}P'''),$$

or

$$(6.7)\qquad S = -4hxR + 2R' - 60h^2xP + 30hP' + 40h^3x^3P - 60h^2x^2P' + 30hxP'' - 5P'''.$$

Let $P = C_0x^6 + C_1x^5 + C_2x^4 + C_3x^3 + C_4x^2 + C_5x + C_6$ and $R = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$. Then equation (6.7) becomes

$$\begin{aligned}(6.8)\qquad S = {} & -4hx(A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4) + 2(4A_0x^3 + 3A_1x^2 \\ & + 2A_2x + A_3) - 60h^2x(C_0x^6 + C_1x^5 + C_2x^4 + C_3x^3 + C_4x^2 \\ & + C_5x + C_6) + 30h(6C_0x^5 + 5C_1x^4 + 4C_2x^3 + 3C_3x^2 + 2C_4x \\ & + C_5) + 40h^3x^3(C_0x^6 + C_1x^5 + C_2x^4 + C_3x^3 + C_4x^2 + C_5x + C_6) \\ & - 60h^2x^2(6C_0x^5 + 5C_1x^4 + 4C_2x^3 + 3C_3x^2 + 2C_4x + C_5) \\ & + 30hx(30C_0x^4 + 20C_1x^3 + 12C_2x^2 + 6C_3x + 2C_4) \\ & - 5(120C_0x^3 + 60C_1x^2 + 24C_2x + 6C_3).\end{aligned}$$

Since S is a polynomial of degree three, at most, the coefficients of x^9, x^8, x^7, x^6, x^5, and x^4 in the right member of equation (6.8) must vanish. If the coefficient of x^9 vanishes, then $40h^3C_0 = 0$, which implies $C_0 = 0$ since $h > 0$. If the coefficient of x^8 vanishes, then $40h^3C_1 = 0$, which implies $C_1 = 0$ since $h > 0$ If the coefficient of x^7 vanishes, then $40h^3C_2 = 0$, which implies $C_2 = 0$. If the coefficient of x^6 vanishes, then $40h^3C_3 = 0$, which implies $C_3 = 0$. If the coefficient of x^5 vanishes, then $-4hA_0 + 40h^3C_4 = 0$, which implies $A_0 = 10h^2C_4$. If the coefficient of x^4 vanishes, then $-4hA_1 + 40h^3C_5 = 0$, which implies $A_1 = 10h^2C_5$.

Hence

$$(6.9)\quad P = C_4x^2 + C_5x + C_6,$$

$$(6.10)\quad R = 10h^2C_4x^4 + 10h^2C_5x^3 + A_2x^2 + A_3x + A_4.$$

The coefficient S is now defined by equation (6.7) with the expressions for P and R taken from equations (6.9) and (6.10).

Orthogonality condition (viii) requires

$$e^{-hx^2}U = (e^{-hx^2}T)' - (e^{-hx^2}R)''' + 3(e^{-hx^2}P)^{v}$$

$$= e^{-hx^2}[-2hxT + T' - 12h^2xR + 6hR' + 8h^3x^3R$$

$$- 12h^2x^2R' + 6hxR'' - R''' + 3(-120h^3xP + 60h^2P'$$

$$+ 180h^4x^3P - 240h^3x^2P' + 120h^2xP'' - 32h^5x^5P$$

$$+ 80h^4x^4P' - 80h^3x^3P'')],$$

or

$$(6.11)\quad U = -2hxT + T' - 12h^2xR + 6hR' + 8h^3x^3R - 12h^2x^2R' + 6hxR''$$

$$- R''' - 360h^3xP + 180h^2P' + 480h^4x^3P - 720h^3x^2P'$$

$$+ 360h^2xP'' - 96h^5x^5P + 240h^4x^4P' - 240h^3x^3P''.$$

Let $T = B_0x^2 + B_1x + B_2$. Then with P and R as defined by equations (6.9) and (6.10), equation (6.11) becomes

$$(6.12)\quad U = -2hx(B_0x^2 + B_1x + B_2) + (2B_0x + B_1)$$

$$- 12h^2x(10h^2C_4x^4 + 10h^2C_5x^3 + A_2x^2 + A_3x + A_4)$$

$$+ 6h(40h^2C_4x^3 + 30h^2C_5x^2 + 2A_2x + A_3)$$

$$+ 8h^3x^3(10h^2C_4x^4 + 10h^2C_5x^3 + A_2x^2 + A_3x + A_4)$$

$$- 12h^2x^2(40h^2C_4x^3 + 30h^2C_5x^2 + 2A_2x + A_3)$$

$$+ 6hx(120h^2C_4x^2 + 60h^2C_5x + 2A_2) - (240h^2C_4x + 60h^2C_5)$$

$$- 360h^3x(C_4x^2 + C_5x + C_6) + 180h^2(2C_4x + C_5)$$

$$+ 480h^4x^3(C_4x^2 + C_5x + C_6) - 720h^3x^2(2C_4x + C_5)$$

$$+ 360h^2x(2C_4) - 96h^5x^5(C_4x^2 + C_5x + C_6)$$

$$+ 240h^4x^4(2C_4x + C_5) - 240h^3x^3(2C_4)$$

Since U is a first degree polynomial, the coefficients of x^7, x^6, x^5, x^4, x^3, and x^2 in the right member of equation (6.12) must vanish. If the coefficient of x^7 vanishes, then $80C_4h^5 - 96C_4h^5 = -16C_4h^5 = 0$, which implies $C_4 = 0$ since $h > 0$. If the coefficient of x^6 vanishes, then $80C_5h^5 - 96C_5h^5 = -16C_5h^5 = 0$, which implies $C_5 = 0$. If the coefficient of x^5 vanishes, then $8A_2h^3 - 96C_6h^5 = 0$, which implies $A_2 = 12C_6h^2$. If the coefficient of x^4 vanishes, then $8A_3h^3 = 0$, which implies $A_3 = 0$. If the coefficient of x^3 vanishes, then $-2B_0h - 12A_2h^2 + 8A_4h^3 - 24A_2h^2 + 480C_6h^4 = 0$, or $B_0 = 4A_4h^2 + 240C_6h^3 - 18A_2h = 4A_4h^2 + 240C_6h^3 - 18(12C_6h^2)h$, or $B_0 = 4A_4h^2 + 24C_6h^3$. Finally, if the coefficient of x^2 vanishes, then $-2B_1h = 0$, which implies $B_1 = 0$. Hence, equations (6.9) and (6.10) become, respectively,

(6.13) $$P = C_6 \neq 0,$$

(6.14) $$R = 12C_6h^2x^2 + A_4.$$

Equation (6.7) can now be written

(6.15) $$S = -4hxR + 2R' - 60C_6h^2x + 40C_6h^3x^3;$$

and equation (6.11) can be written

(6.16) $$U = -2hxT + T' - 12h^2xR + 6hR' + 8h^3x^3R - 12h^2x^2R'$$

$$+ 6hxR'' - 360C_6h^3x + 480C_6h^4x^3 - 96C_6h^5x^5,$$

where

(6.17) $$T = (4A_4h^2 + 24C_6h^3)x^2 + e_2.$$

An expression for Q may now be obtained by application of orthogonality condition (ix), which requires

$$e^{-hx^2}Q = 3(e^{-hx^2}P)' = 3(C_6e^{-hx^2})' = 3C_6(e^{-hx^2})'$$

$$= -6C_6hxe^{-hx^2},$$

or

(6.18) $$Q = -6C_6hx.$$

The remaining orthogonality conditions (i) through (vi) are satisfied at $\pm\infty$ for $w = e^{-hx^2}$, $h > 0$, as will be shown in the following. Since $wP = e^{-hx^2}C_6 = 0$ at $\pm\infty$, condition (i) is satisfied. Since $(wP)' = -2C_6hxe^{-hx^2} = 0$ at $\pm\infty$, condition (ii) is satisfied. Since $(wP)'' = -2C_6he^{-hx^2}(1 - 2hx^2) = 0$ at $\pm\infty$, condition (iii) is satisfied. Since $wR - (wP)'' = e^{-hx^2}[R+2C_6h(1-2hx^2)]=0$

at $\pm\infty$, condition (iv) is satisfied. Since $(wR)' - 3(wP)'''$ $= e^{-hx^2}[(-2hxR + R') - 3(12C_6h^2x - 8C_6h^3x^3)] = 0$ at $\pm\infty$, condition (v) is satisfied. Since $wT - (wR)'' + 3(wP)^{iv} = e^{-h^2x}[T - (-2hR + 4h^2x^2R - 4hxR' + R'') + 3(12C_6h^2 - 48C_6h^3x^2 + 16C_6h^4x^4)] = 0$ at $\pm\infty$, condition (vi) is satisfied.

Equations (6.13) through (6.18) constitute the totality of restrictions imposed on the coefficients P, Q, R, S, T, and U by the nine orthogonality conditions for the infinite interval with $w = e^{-hx^2}$, $h > 0$. In the summary, C_6 will be replaced by K

Summary

The choice of $w(x) = e^{-hx^2}$, where $h > 0$, will serve as a weight function in the orthogonalization of the solution set $\{y_n\}$, $n = 0,1,2,\cdots$, represented by equation (1.2) of the differential equation (1.23) over the interval $(-\infty, \infty)$

This choice of $w(x)$ impose the following definitions of the coefficients of equation (1.23) in the satisfaction of the nine orthogonality conditions.

$P = K$, a non-zero constant.

$Q = -6Khx$

$R = 12Kh^2x^2 + A_4$, A_4 a constant.

$S = -4hxR + 2R' - 80Kh^2x + 40Kh^3x^3$

$T = (4A_4h^2 + 24Kh^3)x^2 + B_2$, B_2 a constant.

$$U = -2hxT + T' - 12h^2xR + 6hR' + 8h^3x^3R$$

$$- 12h^2x^2R' + 6hxR'' - 360Kh^3x + 480Kh^4x^3$$

$$- 96Kh^5x^5.$$

Example of the Infinite Interval

Choose $h = 1$. Then

(6.E1) $w = e^{-x^2}$,

(6.E2) $P = K \neq 0$,

(6.E3) $Q = -6Kx$,

(6.E4) $R = 12Kx^2 + A_4$,

(6.E5)
$$S = -4x(12Kx^2 + A_4) + 2(24Kx) - 60Kx + 40Kx^3$$
$$= -8Kx^3 - 4(A_4 + 3K)x,$$

(6.E6)
$$T = (4A_4 + 24K)x^2 + B_2$$
$$= 4(A_4 + 6K)x^2 + B_2,$$

(6.E7)
$$U = -8(A_4 + 6K)x^3 - 2B_2x + 6(A_4 + 6K)x$$
$$- 12x(12Kx^2 + A_4) + 6(24Kx) + 8x^3(12Kx^2 + A_4)$$
$$- 12x^2(24Kx) + 6x(24K) - 360Kx + 480Kx^3 - 96Kx^5$$
$$= -2(2A_4 + B_2 + 12K)x$$

The differential equation is

(6.E8)
$$Ky_n^{vi} - 6Kxy_n^{v} + (12Kx^2 + A_4)y_n^{iv} - [8Kx^3 + 4(A_4 + 3K)x]y_n'''$$
$$+ [4(A_4 + 6K)x^2 + B_2]y_n'' - 2(2A_4 + B_2 + 12K)xy_n' + \lambda_n y_n = 0.$$

Solutions of the form

$$y_n = \sum_{j=0}^{n} a_{jn} x^{n-j}, \qquad a_{on} \neq 0, \tag{6.E9}$$

will be obtained.

$$\begin{array}{r|l}
\lambda_n & y_n = a_{on}x^n + \cdots + a_{nn}, \quad a_{on} \neq 0 \\
(-4A_4 - 2B_2 - 24K)x & y_n' = na_{on}x^{n-1} + \cdots + a_{n-1,n} \\
4(A_4 + 6K)x^2 + B_2 & y_n'' = n(n-1)a_{on}x^{n-2} + \cdots + 2a_{n-2,n} \\
-8Kx^3 - 4(A_4 + 3K)x & y_n''' = n(n-1)(n-2)a_{on}x^{n-3} + \cdots + 6a_{n-3,n} \\
12Kx^2 + A_4 & y_n^{iv} = n(n-1)(n-2)(n-3)a_{on}x^{n-4} + \cdots + 24a_{n-4,n} \\
-6Kx & y_n^{v} = n(n-1)(n-2)(n-3)(n-4)a_{on}x^{n-5} + \cdots + 120a_{n-5,n} \\
K & y_n^{vi} = n(n-1)(n-2)(n-3)(n-4)(n-5)a_{on}x^{n-6} + \cdots + 720a_{n-6,n} \\
\hline
\end{array}$$

The highest power of x which appears is n. The coefficient of x^n is

$$[\lambda_n - (4A_4 + 2B_2 + 24K)n + 4(A_4 + 6K)n(n-1) - 8Kn(n-1)(n-2)]a_{on} = 0.$$

Since $a_{on} \neq 0$,

$$\begin{aligned} \lambda_n &= n(4A_4 + 2B_2 + 24K - 4A_4 n - 24Kn + 4A_4 + 24K \\ &\quad + 8Kn^2 - 24Kn + 16K) \\ &= 2n[4Kn^2 - 2(A_4 + 12K)n + (4A_4 + B_2 + 32K)]. \end{aligned} \tag{6.E10}$$

The choices of K, A_4, and B_2 are arbitrary, with the exception that $K \neq 0$. Hence, choose $K = \frac{1}{4}$, $A_4 = -2$, $B_2 = 0$, and thus simplify λ_n to

$$\lambda_n = 2n(n^2 - 2n) = 2n^2(n-2). \tag{6.E11}$$

Division of equation (6.E8) by K and substitution of $K = \frac{1}{4}$, $A_4 = -2$, and $B_2 = 0$ in the result yields

$$y_n^{vi} - 6xy_n^{v} + (12x^2 - 8)y_n^{iv} - (8x^3 - 20x)y_n''' - 6x^2y_n'' + 0xy_n' + l_ny_n = 0, \tag{6.E12}$$

where

$$l_n = \frac{\lambda_n}{K} = 4\lambda_n = 8n^2(n-2). \tag{6.E13}$$

For each choice of n, $n = 0,1,2, \cdots$, the value of l_n is determined from equation (6.E13), and a polynomial solution of degree n of equation (6.E12) can be obtained. The determinations of a few of these solutions follow

Let $n = 0$. Then

$$y_0 = a_{00}, \tag{6.E14}$$

$$y_0' = y_0'' = y_0''' = y_0^{iv} = y_0^{v} = y_0^{vi} = l_0 = 0,$$

so that equation (6.E12) becomes

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary.

<u>Let $n = 1$.</u> Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{iv} = y_1^{v} = y_1^{vi} = 0,$$

$$l_1 = 8(1)(-1) = -8,$$

so that equation (8.E12) becomes

$$8a_{01}x - 8(a_{01}x + a_{11}) = 0 \cdot a_{01}x - 8a_{11} = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = 0$. Consequently,

$$y_1 = a_{01}x. \tag{8.E15}$$

<u>Let $n = 2$.</u> Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{iv} = y_2^{v} = y_2^{vi} = 0,$$

$$l_2 = 8(4)(0) = 0,$$

so that equation (8.E12) becomes

$$-16a_{02}x^2 + 8x(2a_{02}x + a_{12}) + 0(a_{02}x^2 + a_{12}x + a_{22})$$

$$= 0 \cdot a_{02}x^2 + 8a_{12}x + 0 \cdot a_{22} = 0.$$

Thus, a_{o2} is arbitrary, $a_{12} = 0$, a_{22} is arbitrary. Consequently,

(6.E16) $$y_2 = a_{o2}x^2 + a_{22}.$$

Let $n = 3$. Then

$$y_3 = a_{o3}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{o3}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{o3}x + 2a_{13},$$

$$y_3''' = 6a_{o3},$$

$$y_3^{iv} = y_3^{v} = y_3^{vi} = 0,$$

$$l_3 = 8(9)(1) = 72.$$

Equation of like powers of x in the differential equation (6.E12) yields

x^3: $-48a_{o3} - 48a_{o3} + 24a_{o3} + 72a_{o3} = 0 \cdot a_{o3} = 0,$

x^2: $-18a_{13} + 18a_{13} + 72a_{13} = 72a_{13} = 0,$

x : $120a_{o3} + 8a_{23} + 72a_{23} = 120a_{o3} + 80a_{23} = 0,$

x^0: $72a_{33} = 0.$

Thus, a_{o3} is arbitrary, $a_{13} = 0$, $a_{23} = -3a_{o3}/2$, $a_{33} = 0$. Hence

(6.E17) $$y_3 = a_{o3}(x^3 - \frac{3}{2}x).$$

Let $n = 4$. Then

$$y_4 = a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44},$$

$$y_4' = 4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34},$$

$$y_4'' = 12a_{04}x^2 + 6a_{14}x + 2a_{24},$$

$$y_4''' = 24a_{04}x + 6a_{14},$$

$$y_4^{iv} = 24a_{04},$$

$$y^{v} = y^{vi} = 0,$$

$$\lambda_4 = 8(16)(2) = 256.$$

Equation of like powers of x in equation (6.E12) yields

$$x^4: \quad -192a_{04} - 96a_{04} + 32a_{04} + 256a_{04} = 0 \cdot a_{04} = 0,$$

$$x^3: \quad -48a_{14} - 48a_{14} + 24a_{14} + 256a_{14} = 184a_{14} = 0,$$

$$x^2: \quad 288a_{04} + 480a_{04} - 16a_{24} + 16a_{24} + 256a_{24}$$

$$= 768a_{04} + 256a_{24} = 0,$$

$$x: \quad 120a_{14} + 8a_{34} + 256a_{34} = 120a_{14} + 264a_{34} = 0,$$

$$x^0: \quad -192a_{04} + 256a_{44} = 0.$$

Thus, a_{04} is arbitrary, $a_{14} = 0$, $a_{24} = -3a_{04}$, $a_{34} = 0$, $a_{44} = 3a_{04}/4$. Consequently,

$$y_4 = a_{04}\left(x^4 - 3x^2 + \frac{3}{4}\right). \qquad (6.E18)$$

Further solutions y_n, $n = 5,6,7, \cdots$, can be determined in a similar manner.

The solution set obtained forms an orthogonal system, with respect to the weight function e^{-x^2}, over the interval $(-\infty, \infty)$. This set might be considered as analogous to a set of Hermite polynomials.

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BIOGRAPHICAL SKETCH

Donald Lee Duncan was born June 7, 1930, at Farmington, Missouri. In June, 1948, Mr. Duncan graduated from Flat River High School in Flat River, Missouri. In June, 1952, he received, with First Honors, the degree of Bachelor of Science in Chemical Engineering from the University of Missouri School of Mines and Metallurgy. From then until September, 1953, he was employed by Union Carbide Nuclear Company at Oak Ridge, Tennessee. From September, 1953, until June, 1956, the author served in the Chemical Corps of the United States Army. Following his discharge from the army, he resumed his academic training in the Graduate School of the University of Missouri, where in August, 1957, he received the degree of Master of Science in Chemical Engineering. While pursuing his graduate studies in engineering he was employed as a teaching assistant in the Department of Mathematics. During the following two years Mr. Duncan worked for Najay Chemical Company in New York, New York, and in Charlotte, North Carolina. In 1959 he entered the Graduate School of the University of Florida and worked as a graduate assistant and then as an instructor in the Department of Mathematics while pursuing his studies leading to the degree of Doctor of Philosophy.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy

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